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Multiloop Amplitudes of Light-cone Gauge Bosonic String Field Theory in Noncritical Dimensions

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Abstract

We study the multiloop amplitudes of the light-cone gauge closed bosonic string field theory for $d \neq 26$. We show that the amplitudes can be recast into a BRST invariant form by adding a nonstandard worldsheet theory for the longitudinal variables $X^{\pm}$ and the reparametrization ghost system. The results obtained in this paper for bosonic strings provide a first step towards the examination whether the dimensional regularization works for the multiloop amplitudes of the light-cone gauge superstring field theory.

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1 Introduction

Light-cone gauge string field theory [1, 2, 3, 4] is a formulation of string theory in which the unitarity of S-matrix is manifest. In the light-cone gauge superstring field theory, regularization is necessary to deal with the contact term problem [5, 6, 7, 8, 9]. In Refs. [10, 11, 12, 13, 14, 15], it has been proposed to employ dimensional regularization to deal with the divergences in string field theory. Being a completely gauge fixed formulation, the light-cone gauge NSR string field theory can be defined in d ($d \neq 10$) dimensions. By taking d to be a large negative value, the divergences of the amplitudes are regularized.[1] Defining the amplitudes as analytic functions of d and taking the $d \to 10$ limit, we obtain the amplitudes in critical dimensions. So far, it has been verified that this dimensional regularization scheme works for the closed string tree-level amplitudes and it has been shown that no divergences occur in the limit $d \to 10$. This implies that at least at tree level we need not add contact interaction terms to the string field theory action as counter terms.

It is obvious that what we should do next is to check whether this regularization scheme works for the multiloop amplitudes as well. In this paper, as a first step towards this goal, we study the multiloop amplitudes of the light-cone gauge closed bosonic string field theory in noncritical dimensions. We evaluate the multiloop amplitudes and give them as integrals over moduli space of the Riemann surfaces corresponding to the Feynman diagrams for strings. We show that they can be rewritten into a BRST invariant form using the conformal gauge worldsheet theory. This can be accomplished by adding the longitudinal variables $X^{\pm}$ and the ordinary reparametrization bc ghosts in a similar way to the case of the tree-level amplitudes [11]. The worldsheet theory for the longitudinal variables is the conformal field theory formulated in Ref. [11], which we refer to as the $X^{\pm}$ CFT.

The organization of this paper is as follows. In section 2, we consider the h-loop N-string amplitudes for the light-cone gauge closed bosonic string field theory in noncritical dimensions. Such an amplitude corresponds to a light-cone string diagram which is conformally equivalent to an N punctured genus h Riemann surface. We present an expression of the amplitude as an integral over the moduli space of the Riemann surface. In section 3, the $X^{\pm}$ CFT on the higher genus Riemann surfaces is constructed. We find that the prescription developed in the sphere case [11] can be directly generalized to the present case. Introducing

[1] Precisely speaking, in order to deal with the amplitudes involving the strings in the (R,NS) and the (NS,R) sectors, we shift the Virasoro central charge of the system, rather than the spacetime dimensions themselves [15]. This can be achieved by adding an extra conformal field theory with sufficiently large negative central charge.

the reparametrization ghost variables as well as the $X^{\pm}$ CFT, we rewrite amplitudes into those of the BRST invariant formulation of strings for $d \neq 26$ in the conformal gauge. Section 4 is devoted to summary and discussions. In appendix A, the definitions of the theta functions, the prime form and the Arakelov Green's functions are presented. In appendix B, the partition functions of the worldsheet theory on the string diagrams are evaluated. In appendix C, we show that the amplitudes of the light-cone gauge string theory are modular invariant even in noncritical dimensions. In appendix D, we present a derivation of an identity which is necessary in section 3 to rewrite the amplitudes into a BRST invariant form.

2 Amplitudes of light-cone string field theory

The light-cone gauge string field theory is defined even for $d \neq 26$. The action for the closed string field theory takes a simple form consisting of a kinetic term and a cubic interaction term:

$$\begin{aligned} S \;=\; & \int dt \left[\frac{1}{2} \int d1d2 \, \langle R(1,2)|\Phi(t)\rangle_1 \left(i\frac{\partial}{\partial t} - \frac{L_0^{\mathrm{LC}(2)} + \tilde{L}_0^{\mathrm{LC}(2)} - \frac{d-2}{12}}{\alpha_2} \right) |\Phi(t)\rangle_2 \right. \\ & \left. + \frac{g}{6} \int d1d2d3 \, \langle V_3(1,2,3)|\Phi(t)\rangle_1 |\Phi(t)\rangle_2 |\Phi(t)\rangle_3 \right] . \end{aligned} \tag{2.1}$$

Here $|\Phi\rangle_r$ is the string field, dr denotes the integration measure of the momentum zero-modes given by

$$dr = \frac{\alpha_r d\alpha_r}{4\pi} \frac{d^{d-2}p_r}{(2\pi)^{d-2}} \,, \tag{2.2}$$

$\alpha_r = 2p_r^+$ is the string-length and $L_0^{\mathrm{LC}(r)}$, $\tilde{L}_0^{\mathrm{LC}(r)}$ denote the zero-modes of the transverse Virasoro generators for the r-th string. The definitions of the reflector $\langle R(1,2)|$ and the three string vertex $\langle V_3(1,2,3)|$ are presented in appendix A of Ref. [11]. Starting from this action, we can evaluate the amplitudes perturbatively. Each term in the expansion corresponds to a light-cone gauge Feynman diagram for strings. A typical 3-loop 5-string diagram is depicted in Figure 1.

Mandelstam mapping

A Euclideanized h-loop N-string diagram is conformally equivalent to an N punctured genus h Riemann surface Σ. The light-cone diagram consists of cylinders. On each cylinder, one

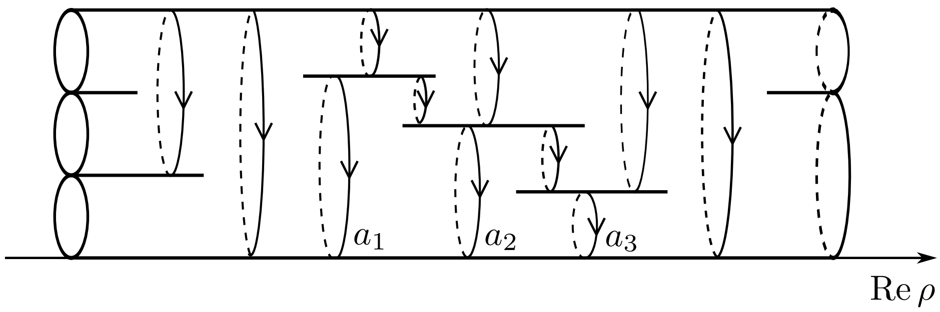

Figure 1: A string diagram with 3 incoming, 2 outgoing strings and 3 loops. The cycles $\mathcal{C}_{\mathcal{I}}$ going around the cylinders corresponding to the internal propagators are described. Among them, the cycle a_j is the a cycle for the j-th loop.

can introduce a complex coordinate ρ whose real part coincides with the Euclideanized light-cone time iX^+ and imaginary part parametrizes the closed string at each time. The ρ's on the cylinders are smoothly connected except at the interaction points and we get a complex coordinate ρ on Σ. ρ is not a good coordinate around the punctures and the interaction points on the light-cone diagram.

ρ can be expressed as an analytic function $\rho(z)$ in terms of a local complex coordinate z on Σ. As in the tree case, $\rho(z)$ is called the Mandelstam mapping. Let Z_r $(r = 1, \ldots, N)$ be the z coordinate of the puncture on Σ which corresponds to the r-th external leg of the light-cone diagram. The Mandelstam mapping $\rho(z)$ can be determined by the two requirements:

1. The one-form $d\rho = \partial\rho(z)dz$ should have simple poles at the punctures Z_r with residues α_r and be non-singular everywhere else.[2]

2. $\mathrm{Re}\,\rho$ has to be globally defined on Σ because it is the light-cone time of the light-cone string diagram. $d\rho$ should therefore have purely imaginary period around any homology cycle.

These two requirements uniquely fix the one-form $d\rho$ [16, 17] and the Mandelstam mapping $\rho(z)$ is obtained as

$$\rho(z) = \sum_{r=1}^{N} \alpha_r \left[\ln E(z, Z_r) - 2\pi i \int_{P_0}^{z} \omega \frac{1}{\mathrm{Im}\,\Omega} \mathrm{Im} \int_{P_0}^{Z_r} \omega \right] , \qquad \sum_{r=1}^{N} \alpha_r = 0 \ , \qquad (2.3)$$

[2]To be precise, $d\rho$ is a meromorphic one-form on the Riemann surface with N marked points Z_r $(r = 1, \ldots, N)$, and holomorphic one-form on the punctured Riemann surface, which has these marked points removed.

up to an additive constant independent of z. Here $\omega = (\omega_j)$ $(j = 1, \ldots, h)$ denotes the canonical basis of the holomorphic one-forms and $\Omega = (\Omega_{jk})$ is the period matrix on Σ, whose definitions are given in appendix A. P_0 is an arbitrary point on Σ, which we take as the base point of the Abel-Jacobi map, and $E(z,w)$ is the prime form [18, 19] defined in eq.(A.8). Since the one-form $d\rho$ has N simple poles, $d\rho$ has $2h-2+N$ simple zeros, which we denote by z_I $(I = 1, \ldots, 2h-2+N)$. They correspond to the interaction points of the light-cone string diagram.

On the light-cone string diagram, the flat metric

$$ds^2 = d\rho d\bar{\rho} \tag{2.4}$$

is chosen as usual, which is referred to as the Mandelstam metric. In terms of a local coordinate z on Σ, it takes the form

$$ds^2 = |\partial\rho(z)|^2 \, dz d\bar{z} \ . \tag{2.5}$$

This metric is singular at the punctures $z = Z_r$ and the interaction points $z = z_I$. For later use, as is done for the tree-level light-cone string diagrams, we introduce the local coordinate w_r around the puncture at $z = Z_r$ defined as

$$w_r(z) = \exp\left[\frac{1}{\alpha_r}\left(\rho(z) - \rho(z_{I^{(r)}})\right)\right] \ , \tag{2.6}$$

where $z_{I^{(r)}}$ denotes the interaction point on the z-plane where the r-th external string interacts.

Amplitudes

It is straightforward to calculate the amplitudes by the old-fashioned perturbation theory starting from the action (2.1) and Euclideanize the time integrals. An h-loop N-string amplitude is given as an integral over the moduli space of the string diagram [20] as

$$\mathcal{A}_N^{(h)} = (ig)^{2h-2+N} C \int [dT][\alpha d\theta][d\alpha] \, F_N^{(h)} \ , \tag{2.7}$$

where $\int [dT][\alpha d\theta][d\alpha]$ denotes the integration over the moduli parameters and C is the combinatorial factor. In each channel, the integration measure is given as

$$\int [dT][\alpha d\theta][d\alpha] = \prod_{a=1}^{2h-3+N} \left(-i \int_0^\infty dT_a\right) \prod_{A=1}^{h} \int \frac{d\alpha_A}{4\pi} \prod_{\mathcal{I}=1}^{3h-3+N} \left(|\alpha_{\mathcal{I}}| \int_0^{2\pi} \frac{d\theta_{\mathcal{I}}}{2\pi}\right) . \tag{2.8}$$

Here T_a's are heights of the cylinders corresponding to internal lines,[3] α_A's denote the string-lengths corresponding to the $+$ components of the loop momenta and $\alpha_{\mathcal{I}}$'s and $\theta_{\mathcal{I}}$'s are the string-lengths and the twist angles for the internal propagators.

The integrand $F_N^{(h)}$ can be described as a correlation function in the light-cone gauge worldsheet theory for the transverse variables X^i $(i = 1, \ldots, d-2)$. It takes the form [11]

$$F_N^{(h)} = (2\pi)^2 \delta\left(\sum_{r=1}^{N} p_r^+\right) \delta\left(\sum_{r=1}^{N} p_r^-\right) \mathrm{sgn}\left(\prod_{r=1}^{N} \alpha_r\right) \left(Z^{\mathrm{LC}}\right)^{\frac{d-2}{24}} \left\langle \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i} . \tag{2.9}$$

Here $V_r^{\mathrm{LC}} \equiv V_r^{\mathrm{LC}}(w_r = 0, \bar{w}_r = 0)$ denotes the light-cone vertex operator for the r-th external leg located at the origin of the local coordinate w_r in eq.(2.6), and $\left\langle \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i}$ is the expectation value of these operators in the worldsheet theory for the transverse variables X^i. $\left(Z^{\mathrm{LC}}\right)^{\frac{d-2}{24}}$ denotes the partition function of the worldsheet theory on the light-cone diagram. The factor $\mathrm{sgn}\left(\prod_{r=1}^{N} \alpha_r\right)$ comes from the peculiar form of the measure of α_r in eq.(2.2) and our convention for the phase of the vertex $\langle V_3|$. The partition function Z^{LC} for $d = 26$ is calculated in appendix B to be

$$Z^{\mathrm{LC}} = \frac{1}{(32\pi^2)^{4h}} e^{-2(h-1)c} e^{2\delta(\Sigma)} \prod_r \left[e^{-2\,\mathrm{Re}\,\bar{N}_{00}^{rr}} (2g^{\mathrm{A}}_{Z_r \bar{Z}_r})^{-1} \alpha_r^{-2} \right] \prod_I \left[\left|\partial^2 \rho(z_I)\right|^{-1} 2g^{\mathrm{A}}_{z_I \bar{z}_I} \right] . \tag{2.10}$$

Here $g^{\mathrm{A}}_{z\bar{z}}$ denotes the Arakelov metric defined in appendix A. The quantities c, $\delta(\Sigma)$ and $\bar{N}_{00}^{rr}$ are defined in eqs.(A.26), (B.44) and (B.18) respectively.

When the r-th external state is of the form

$$\alpha_{-n_1}^{i_1(r)} \cdots \tilde{\alpha}_{-\tilde{n}_1}^{\tilde{i}_1(r)} \cdots \left|p^-, p^i\right\rangle_r \ , \tag{2.11}$$

the vertex operator V_r^{LC} becomes

$$\begin{aligned} V_r^{\mathrm{LC}} &= \alpha_r \oint_0 \frac{dw_r}{2\pi i} i\partial X^{i_1}(w_r) w_r^{-n_1} \cdots \oint_0 \frac{d\bar{w}_r}{2\pi i} i\bar{\partial} X^{\tilde{i}_1}(\bar{w}_r) \bar{w}_r^{-\tilde{n}_1} \cdots \\ &\quad \times e^{-p_r^- \tau_0^{(r)}} e^{i p_r^i X^i} (w_r = 0, \bar{w}_r = 0) \ , \end{aligned} \tag{2.12}$$

in which the operators are normal ordered, w_r denotes the local coordinate introduced in eq.(2.6) and $\tau_0^{(r)} = \mathrm{Re}\,\rho\left(z_{I^{(r)}}\right)$. The on-shell and the level-matching conditions require that

$$\frac{1}{2}\left(-2p_r^+ p_r^- + p_r^i p_r^i\right) + \mathcal{N}_r = \frac{d-2}{24} \ , \qquad \mathcal{N}_r \equiv \sum_j n_j = \sum_j \tilde{n}_j \ . \tag{2.13}$$

[3]Heights of the cylinders in a light-cone diagram are constrained so that only $2h - 3 + N$ of them can be varied independently.

We note that the definition of V_r^{LC} is independent of the choice of the worldsheet metric to define the theory and so is the expectation value $\left\langle \prod_{r=1}^N V_r^{\mathrm{LC}} \right\rangle^{X^i}$. The expectation value $\left\langle \prod_{r=1}^N V_r^{\mathrm{LC}} \right\rangle^{X^i}$ can be expressed in terms of the Green's functions on Σ. Therefore we can express the integrand (2.9) in terms of various quantities on Σ defined in appendix A.

Comments

Before closing this section, several comments are in order.

1. Using the Mandelstam mapping, Z^{LC} can be considered as the partition function on the surface endowed with the Mandelstam metric (2.5) and be written as

$$Z^{\mathrm{LC}} = \left(Z^X \left[\frac{1}{2} |\partial\rho(z)|^2 \right] \right)^{24} , \tag{2.14}$$

where

$$Z^X[g_{z\bar{z}}] \equiv \left(\frac{8\pi^2 \det{}' \triangle_{g_{z\bar{z}}}}{\int dz \wedge d\bar{z} \sqrt{g}} \right)^{-\frac{1}{2}} , \tag{2.15}$$

and $\triangle_{g_{z\bar{z}}} = -2g^{z\bar{z}}\partial_z\partial_{\bar{z}}$ denotes the scalar Laplacian for conformal gauge metric $ds^2 = 2g_{z\bar{z}}dzd\bar{z}$. From the expression (2.14), we get

$$Z^{\mathrm{LC}} = Z^X[g^{\mathrm{A}}_{z\bar{z}}]^{24} e^{-\Gamma\left[g^{\mathrm{A}}_{z\bar{z}},\, \ln|\partial\rho|^2\right]} , \tag{2.16}$$

where $\Gamma\left[g^{\mathrm{A}}_{z\bar{z}},\, \ln|\partial\rho|^2\right]$ is the Liouville action

$$\Gamma\left[g^{\mathrm{A}}_{z\bar{z}},\, \ln|\partial\rho|^2\right] = -\frac{24}{48\pi} \int dz \wedge d\bar{z}\, i \left(\partial\chi\bar{\partial}\chi + g^{\mathrm{A}}_{z\bar{z}} R^{\mathrm{A}} \chi \right) , \tag{2.17}$$

with $\chi(z,\bar{z}) = \ln|\partial\rho(z)|^2 - \ln(2g^{\mathrm{A}}_{z\bar{z}})$.

In Refs. [21, 22], $\Gamma\left[g^{\mathrm{A}}_{z\bar{z}},\, \ln|\partial\rho|^2\right]$ is calculated by directly evaluating the Liouville action (2.17). In order to do so, one needs to regularize the divergences coming from the singularities of the Mandelstam metric (2.5). In Ref. [21] a Weyl invariant but reparametrization noninvariant regularization is employed and $e^{-\Gamma\left[g^{\mathrm{A}}_{z\bar{z}},\, \ln|\partial\rho|^2\right]}$ is evaluated to be

$$e^{-2(h-1)c} \prod_r \left[(2g^{\mathrm{A}}_{Z_r\bar{Z}_r})^{-1} \alpha_r^2 \right] \prod_I \left[\left|\partial^2\rho(z_I)\right|^{-2} 2g^{\mathrm{A}}_{z_I\bar{z}_I} \right] . \tag{2.18}$$

Our result for Z^{LC} implies that $e^{-\Gamma\left[g^{\mathrm{A}}_{z\bar{z}},\, \ln|\partial\rho|^2\right]}$ is the one given in eq.(B.42), which includes extra factors of $e^{-2\,\mathrm{Re}\,\bar{N}^{rr}_{00}}$ and $|\partial^2\rho(z_I)|$ compared with eq.(2.18). These factors make $\Gamma\left[g^{\mathrm{A}}_{z\bar{z}}, \ln|\partial\rho|^2\right]$ both Weyl and reparametrization invariant as it should be.[4] The regularization employed in Ref. [21] does not cause any problems in deriving the equivalence of the light-cone amplitudes and the covariant ones in the critical dimensions, but it is not appropriate for the noncritical case.

2. In the light-cone gauge expression (2.7) of the amplitudes, the moduli parameters T, α, θ are modular invariant, and the integration region covers the moduli space only once [16]. Being a function of these parameters, the integrand $F^{(h)}_N$ should also be invariant under the modular transformations. However, the explicit form (2.10) of Z^{LC} and the correlation function $\left\langle \prod_{r=1}^N V^{\mathrm{LC}}_r \right\rangle^{X^i}$ are given in terms of the quantities such as the theta functions which depend on the choice of the cycles a_j, b_j. As a consistency check, it is possible to show that these quantities are invariant under the modular transformations and do not depend on the choice of these cycles. The details are given in appendix C.

3 BRST invariant form of the amplitudes

In this section, we would like to rewrite the integrand $F^{(h)}_N$ of the amplitudes, given in eq.(2.9), into the correlation function of the worldsheet theory for strings in the conformal gauge. By doing so, we will show that there exists a BRST invariant formulation in the conformal gauge corresponding to the string field theory in the noncritical dimensions. All these have been done for the tree-level amplitudes in Ref. [11].

3.1 $X^\pm$ CFT

In order to get the BRST invariant formulation, we need to introduce the longitudinal variables $X^\pm$ and the reparametrization bc ghosts. The worldsheet theory of $X^\pm$ for $d \neq 26$ is constructed and called $X^\pm$ CFT [11]. In this subsection, we would like to consider the $X^\pm$ CFT on the Riemann surface Σ of genus h.

For $d = 26$, the longitudinal variables $X^\pm$ are introduced in the form of the following path integral:

$$\int \left[dX^+ dX^-\right]_{\hat{g}_{z\bar{z}}} e^{-S^{\pm}_{d=26}} \prod_{r=1}^N e^{-ip^+_r X^-}(Z_r, \bar{Z}_r) \prod_{s=1}^M e^{-ip^-_s X^+}(z_s, \bar{z}_s)\,, \tag{3.1}$$

[4] It will be possible to get such factors by using the more intricate regularization method in Ref. [23].

where

$$S^{\pm}_{d=26} = -\frac{1}{4\pi} \int dz \wedge d\bar{z}\, i \left(\partial X^{+} \bar{\partial} X^{-} + \partial X^{-} \bar{\partial} X^{+} \right) . \tag{3.2}$$

Here we take a worldsheet metric $ds^2 = \hat{g}_{z\bar{z}} dz d\bar{z}$ to define the path integral measure. For $d = 26$, we do not have to worry about the choice of $\hat{g}_{z\bar{z}}$.

Since the action for $X^{\pm}$ is not bounded below, we need to take the integration contour of $X^{\pm}$ carefully to define the integral. We decompose the variable $X^{\pm}$ as

$$X^{\pm}(z, \bar{z}) = X^{\pm}_{\rm cl}(z, \bar{z}) + x^{\pm} + \delta X^{\pm}(z, \bar{z}) , \tag{3.3}$$

where $X^{\pm}_{\rm cl}(z, \bar{z})$ are solutions to the equations of motion with the source terms[5]

$$\begin{aligned} \partial \bar{\partial} X^{+}_{\rm cl}(z, \bar{z}) &= -i \sum_r p^{+}_r (-2\pi i) \delta^2(z - Z_r) , \\ \partial \bar{\partial} X^{-}_{\rm cl}(z, \bar{z}) &= -i \sum_s p^{-}_s (-2\pi i) \delta^2(z - z_s) , \end{aligned} \tag{3.4}$$

and $x^{\pm} + \delta X^{\pm}(z, \bar{z})$ are the fluctuations around the solutions with

$$\int dz \wedge d\bar{z} \sqrt{\hat{g}} \delta X^{\pm} = 0 . \tag{3.5}$$

The integrals over $X^{\pm}$ are expressed as those over $x^{\pm}$ and $\delta X^{\pm}$. In eq.(3.1), we take the integration contours of $x^{\pm}$ and $\delta X^{+} - \delta X^{-}$ to be along the real axis and that of $\delta X^{+} + \delta X^{-}$ to be along the imaginary axis. Then eq.(3.1) becomes well-defined and is evaluated to be

$$(2\pi)^2 \delta \left(\sum_r p^{+}_r \right) \delta \left(\sum_s p^{-}_s \right) \prod_{s=1}^{M} e^{-ip^{-}_s X^{+}_{\rm cl}}(z_s, \bar{z}_s) Z^X [\hat{g}_{z\bar{z}}]^2 . \tag{3.6}$$

Here we can see that the insertion $\prod_{s=1}^{N} e^{-ip^{-}_s X^{+}}(z_s, \bar{z}_s)$ in the path integral (3.1) is replaced by its classical value $\prod_{s=1}^{N} e^{-ip^{-}_s X^{+}_{\rm cl}}(z_s . \bar{z}_s)$. From eq.(3.4), we can take

$$X^{+}_{\rm cl}(z, \bar{z}) = -\frac{i}{2} \left(\rho\left(z\right) + \bar{\rho}\left(\bar{z}\right) \right) , \tag{3.7}$$

the right hand side of which coincides with the the Lorentzian time of the light-cone string diagram. As will be explained in the next subsection, one can relate the DDF vertex operators and the light-cone gauge ones using this fact. Therefore, by considering the path integral (3.1), one can introduce the longitudinal variables essentially satisfying the light-cone gauge conditions. Multiplying the path integral for the transverse variables X^i by the

[5]The delta function $\delta^2(z)$ in eq.(3.4) is normalized so that $\int dz \wedge d\bar{z}\, \delta^2(z) = 1$.

path integral (3.1) for $X^{\pm}$ and that for the bc ghosts, we are able to get the path integral for the conformal gauge worldsheet theory.

For $d \neq 26$, we should specify the metric on the worldsheet to define the path integral. In the light-cone gauge formulation, the natural choice is

$$ds^2 = -4\partial X^+ \bar{\partial} X^+ dz d\bar{z} \,. \tag{3.8}$$

This choice is meaningful only when $\partial X^+ \bar{\partial} X^+$ possesses a reasonable expectation value. In our case, the expectation value coincides with the Mandelstam metric (2.5). The path integral measure defined with such a metric can be given as

$$\left[dX^i dX^{\pm} db d\tilde{b} dc d\tilde{c}\right]_{-4\partial X^+ \bar{\partial} X^+} = \left[dX^i dX^{\pm} db d\tilde{b} dc d\tilde{c}\right]_{\hat{g}_{z\bar{z}}} e^{-\frac{d-26}{24}\Gamma\left[\hat{g}_{z\bar{z}}, \ln\left(-4\partial X^+ \bar{\partial} X^+\right)\right]} \,. \tag{3.9}$$

Thus an extra dependence on the variable X^+ comes from the path integral measure. Including it into the action for $X^{\pm}$ variables, the action $S^{\pm}$ becomes

$$S^{\pm}\left[\hat{g}_{z\bar{z}}\right] = -\frac{1}{4\pi}\int dz \wedge d\bar{z}\, i\left(\partial X^+ \bar{\partial} X^- + \partial X^- \bar{\partial} X^+\right) + \frac{d-26}{24}\Gamma\left[\hat{g}_{z\bar{z}}, \ln\left(-4\partial X^+ \bar{\partial} X^+\right)\right] \,. \tag{3.10}$$

In order to relate the light-cone gauge formulation and the conformal gauge one, we should consider

$$\int \left[dX^+ dX^-\right]_{\hat{g}_{z\bar{z}}} e^{-S^{\pm}\left[\hat{g}_{z\bar{z}}\right]} \prod_{r=1}^{N} e^{-ip_r^+ X^-}(Z_r, \bar{Z}_r) \prod_{s=1}^{M} e^{-ip_s^- X^+}(z_s, \bar{z}_s) \,. \tag{3.11}$$

This path integral can be evaluated easily as follows. Using the prescription for the contour of the integration, it is straightforward to prove

$$\begin{aligned} 0 \;=\; & \int \left[dX^+ dX^-\right]_{\hat{g}_{z\bar{z}}} e^{-S^{\pm}_{d=26}} \\ & \times \prod_{r=1}^{N} e^{-ip_r^+ X^-}(Z_r.\bar{Z}_r) \prod_{s=1}^{M} e^{-ip_s^- X^+}(z_s.\bar{z}_s) \prod_p \partial\delta X^+(z_p) \prod_q \bar{\partial}\delta X^+(\bar{z}_q) \,. \end{aligned} \tag{3.12}$$

It follows that substituting $X^+ = X^+_{\mathrm{cl}} + x^+ + \delta X^+$ into the term $\Gamma\left[\hat{g}_{z\bar{z}}, \ln\left(-4\partial X^+ \bar{\partial} X^+\right)\right]$ of the action $S^{\pm}$ in eq.(3.11) and expanding it in terms of δX^+, we get

$$\begin{aligned} & \left\langle \prod_{r=1}^{N} e^{-ip_r^+ X^-}(Z_r.\bar{Z}_r) \prod_{s=1}^{M} e^{-ip_s^- X^+}(z_s.\bar{z}_s) \right\rangle^{X^{\pm}}_{\hat{g}_{z\bar{z}}} \\ & \quad \equiv Z^X\left[\hat{g}_{z\bar{z}}\right]^{-2} \int \left[dX^+ dX^-\right]_{\hat{g}_{z\bar{z}}} e^{-S^{\pm}\left[\hat{g}_{z\bar{z}}\right]} \prod_{r=1}^{N} e^{-ip_r^+ X^-}(Z_r, \bar{Z}_r) \prod_{s=1}^{M} e^{-ip_s^- X^+}(z_s, \bar{z}_s) \\ & \quad = (2\pi)^2 \delta\left(\sum_s p_s^-\right) \delta\left(\sum_r p_r^+\right) \prod_s e^{-p_s^- \frac{\rho+\bar{\rho}}{2}}(z_s, \bar{z}_s)\, e^{-\frac{d-26}{24}\Gamma\left[\hat{g}_{z\bar{z}}, \ln|\partial\rho|^2\right]} \,. \end{aligned} \tag{3.13}$$

Taking $\Gamma\left[\hat{g}_{z\bar{z}}, \ln|\partial\rho|^2\right]$ to be the one given in eq.(B.40), it is possible to calculate various correlation functions of $X^\pm$ from eq.(3.13). One can show that the energy-momentum tensor of the $X^\pm$ CFT satisfies the Virasoro algebra with the central charge $28-d$ [11]. Therefore the worldsheet theory for $X^\pm, X^i, b, c, \tilde{b}, \tilde{c}$ is a CFT with vanishing central charge.

3.2 BRST invariant form of amplitudes

Using eq.(3.13), we find that the product of the light-cone vertex operators V_r^{LC} each of which corresponds to the light-cone state (2.11) is expressed as the expectation value of that of the DDF vertex operators in the $X^\pm$ CFT:

$$
\begin{aligned}
&(2\pi)^2\delta\left(\sum_{r=1}^N p_r^+\right)\delta\left(\sum_{r=1}^N p_r^-\right)\prod_{r=1}^N V_r^{\mathrm{LC}} \\
&= \prod_{r=1}^N\left(\alpha_r e^{2\,\mathrm{Re}\,\bar{N}_{00}^{rr}}\right) e^{\frac{d-26}{24}\Gamma\left[\hat{g}_{z\bar{z}},\,\ln|\partial\rho|^2\right]}\left\langle\prod_{r=1}^N\left[V_r^{\mathrm{DDF}}(Z_r,\bar{Z}_r)e^{\frac{d-26}{24}\frac{i}{p_r^+}X^+}(z_{I^{(r)}},\bar{z}_{I^{(r)}})\right]\right\rangle_{\hat{g}_{z\bar{z}}}^{X^\pm} .
\end{aligned}
\tag{3.14}
$$

Here V_r^{DDF} is the DDF vertex operator given by

$$
V_r^{\mathrm{DDF}}(z,\bar{z}) = A_{-n_1}^{i_1(r)}(z)\cdots\tilde{A}_{-\tilde{n}_1}^{\tilde{\imath}_1(r)}(\bar{z})\cdots e^{-ip_r^+X^- - i\left(p_r^- - \frac{\mathcal{N}_r}{p_r^+}+\frac{d-26}{24}\frac{1}{p_r^+}\right)X^+ + ip_r^iX^i}(z,\bar{z})\ , \tag{3.15}
$$

with the DDF operator $A_{-n}^{i(r)}$ for the r-th string defined as

$$
A_{-n}^{i(r)}(z) = \oint_z \frac{dz'}{2\pi i}\, i\partial X^i(z') e^{-i\frac{n}{p_r^+}X_L^+(z')}\ , \tag{3.16}
$$

and $\tilde{A}_{-\tilde{n}}^{\tilde{\imath}(r)}$ similarly given for the anti-holomorphic sector. The operators in eq.(3.15) are normal ordered, and $X_L^+(z)$ in eq.(3.16) denotes the holomorphic part of $X^+(z,\bar{z})$.

Substituting eq.(3.14) into eq.(2.9) with the worldsheet metric $\hat{g}_{z\bar{z}}$ taken to be $g_{z\bar{z}}^{\mathrm{A}}$, we find that the integrand $F_N^{(h)}$ of the amplitude (2.7) is expressed as

$$
\begin{aligned}
F_N^{(h)} \ \propto\ & \prod_{r=1}^N\left(\alpha_r e^{2\,\mathrm{Re}\,\bar{N}_{00}^{rr}}\right) e^{-\Gamma\left[g_{z\bar{z}}^{\mathrm{A}},\,\ln|\partial\rho|^2\right]} Z^X[g_{z\bar{z}}^{\mathrm{A}}]^{-2} \\
& \times Z^X[g_{z\bar{z}}^{\mathrm{A}}]^d\left\langle\prod_{r=1}^N\left[V_r^{\mathrm{DDF}}(Z_r,\bar{Z}_r)e^{\frac{d-26}{24}\frac{i}{p_r^+}X^+}(z_{I^{(r)}},\bar{z}_{I^{(r)}})\right]\right\rangle_{g_{z\bar{z}}^{\mathrm{A}}}^{X^\mu} ,
\end{aligned}
\tag{3.17}
$$

where $\langle\cdots\rangle_{\hat{g}_{z\bar{z}}}^{X^\mu}$ denotes the correlation function in the combined system of the worldsheet theory for X^i $(i=1,\ldots,d-2)$ and the $X^\pm$ CFT with the metric $\hat{g}_{z\bar{z}}$.

We will further rewrite eq.(3.17) by introducing the ghost variables. It is possible to show the following identity:

$$
\begin{aligned}
&\prod_{r=1}^{N}\left(\alpha_r e^{2\,\mathrm{Re}\,\bar{N}_{00}^{rr}}\right) e^{-\Gamma\left[g_{z\bar{z}}^{\mathrm{A}},\,\ln|\partial\rho|^2\right]} Z^X[g_{z\bar{z}}^{\mathrm{A}}]^{-2} \\
&= \mathrm{const.}\int \left[dbd\tilde{b}dcd\tilde{c}\right]_{g_{z\bar{z}}^{\mathrm{A}}} e^{-S^{bc}} \prod_{r=1}^{N} c\tilde{c}(Z_r,\bar{Z}_r) \prod_{K=1}^{6h-6+2N}\left[\int dz\wedge d\bar{z}\, i\left(\mu_K b+\bar{\mu}_K\tilde{b}\right)\right]. \quad (3.18)
\end{aligned}
$$

Here S^{bc} is the action for the bc ghosts, μ_K $(K=1,\cdots,6h-6+2N)$ denote the Beltrami differentials for the moduli parameters T,α,θ, and const. indicates a constant independent of the moduli parameters. Eq.(3.18) is derived in appendix D.

Substituting eq.(3.18) into eq.(3.17), we eventually get

$$
\begin{aligned}
\mathcal{A}_N^{(h)} \sim & \int [dT][d\alpha][\alpha d\theta] \\
& \times \int \left[dX^\mu dbd\tilde{b}dcd\tilde{c}\right]_{g_{z\bar{z}}^{\mathrm{A}}} e^{-S^{X^i}-S^{\pm}[g_{z\bar{z}}^{\mathrm{A}}]-S^{bc}} \prod_{K=1}^{6h-6+2N}\left[\int dz\wedge d\bar{z}\, i\left(\mu_K b+\bar{\mu}_K\tilde{b}\right)\right] \\
& \times \prod_{r=1}^{N}\left[c\tilde{c}V_r^{\mathrm{DDF}}(Z_r,\bar{Z}_r) e^{\frac{d-26}{24}\frac{i}{p_r^+}X^+}(z_{I(r)},\bar{z}_{I(r)})\right], \quad (3.19)
\end{aligned}
$$

where S^{X^i} denotes the action for the worldsheet theory of the transverse variables X^i. This form of the amplitude is BRST invariant. The vertex operators $c\tilde{c}V_r^{\mathrm{DDF}}$ are BRST invariant on the mass shell, the insertions $e^{\frac{d-26}{24}\frac{i}{p_r^+}X^+}$ at interaction points are BRST invariant [11] and the BRST variation of the antighost insertion yields the total derivative with respect to the moduli parameter. Eq.(3.19) is a generalization of the results in Refs. [20, 21] to the noncritical case.

4 Summary and discussions

In this paper, we have studied the multiloop amplitudes of the light-cone gauge bosonic string field theory in noncritical dimensions. The amplitudes are expressed as integrals over the moduli space of the Riemann surfaces corresponding to the light-cone diagrams. We have constructed the worldsheet theory for the $X^\pm$ variables on the higher genus surfaces. It has been shown that the multiloop amplitudes of the light-cone string theory in noncritical dimensions can be rewritten into BRST invariant ones of the conformal gauge worldsheet theory consisting of the theory for the transverse variables X^i, the $X^\pm$ CFT and the reparametrization bc ghost system.

We can at least formally construct the gauge invariant string field theory based on the conformal gauge worldsheet theory mentioned above. Since we have constructed the CFT on the worldsheet of the light-cone string diagram, this string field theory should possess the joining-splitting type interaction. Such a theory is expected to be a version formulated in noncritical dimensions of the $\alpha = p^+$ HIKKO theory [24, 25, 26].

Since the on-shell condition is given by eq.(2.13), we can regularize the infrared behavior of the amplitudes by taking d to be negative and large. The divergences of multiloop amplitudes of string theory are infrared divergences and we expect that our formulation works as a regularization of them. The expression for the multiloop amplitudes in the light-cone gauge formalism will not be so useful for practical calculations. We need to know the coordinates z_I of the interaction points to evaluate the amplitudes, which is technically impossible for almost all the cases. Moreover, the parametrization of the moduli space is not holomorphic. What we intend to do is not finding a way of calculations, but showing that the amplitudes can be deduced from a simple string field theory action with only cubic interaction term.

Bosonic string theory itself is not so interesting anyway, because of the existence of tachyon. In order to discuss tachyon free theory, we need to supersymmetrize the analyses in this paper and investigate whether the dimensional regularization scheme proposed in Refs. [10, 11, 12, 13, 14, 15] works for the multiloop amplitudes in the light-cone gauge NSR superstring field theory. Using the supersheet technique [27, 28, 29, 30], it will be possible to relate the results of the light-cone gauge formalism to those of the covariant formalism using the super Riemann surfaces [31, 32, 33, 34].

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A Theta functions, prime form and Arakelov Green's function

In this appendix, we explain various quantities defined on a genus h Riemann surface Σ, which are necessary to express the multiloop amplitudes of the light-cone gauge string field theory.[6]

In the usual way, we choose on Σ a canonical basis $\{a_j, b_j\}$ ($j = 1, \ldots, h$) of homology cycles. Let $\omega = (\omega_j) = (\omega_1, \ldots, \omega_h)$ be the dual basis of holomorphic one-forms on Σ:

$$\oint_{a_j} \omega_k = \delta_{jk} \ , \quad \oint_{b_j} \omega_k = \Omega_{jk} \ , \tag{A.1}$$

where $\Omega = (\Omega_{jk})$ is the period matrix, which is a symmetric $h \times h$ complex matrix with positive definite imaginary part, $\operatorname{Im}\Omega > 0$.

Theta functions

With the period matrix Ω in eq.(A.1), any point $\delta \in \mathbb{C}^h$ can be uniquely expressed in terms of two $\mathbb{R}^h$-vectors as

$$\delta = \delta'\Omega + \delta'' \ , \qquad \delta', \delta'' \in \mathbb{R}^h \ . \tag{A.2}$$

The notation $[\delta] = \left[\begin{smallmatrix}\delta' \\ \delta''\end{smallmatrix}\right]$ is used to represent the point $\delta \in \mathbb{C}^h$ in eq.(A.2). The theta function with characteristics $[\delta] = \left[\begin{smallmatrix}\delta' \\ \delta''\end{smallmatrix}\right]$ is defined by

$$\begin{aligned}
\theta[\delta](\zeta|\Omega) &= \sum_{n\in\mathbb{Z}^h} e^{2\pi i\left[\frac{1}{2}(n+\delta')\Omega(n+\delta')+(n+\delta')(\zeta+\delta'')\right]} \\
&= e^{2\pi i\left[\frac{1}{2}\delta'\Omega\delta'+\delta'(\zeta+\delta'')\right]}\,\theta(\zeta+\delta''+\delta'\Omega|\Omega) \ ,
\end{aligned} \tag{A.3}$$

where $\theta(\zeta|\Omega) = \theta[0](\zeta|\Omega)$. $\theta[\delta](\zeta|\Omega)$ is a quasi-periodic function on the Jacobian variety $J(\Sigma) = \mathbb{C}^h/(\mathbb{Z}^h + \mathbb{Z}^h\Omega)$ of the Riemann surface Σ and transforms as

$$\theta[\delta](\zeta+m+n\Omega|\Omega) = e^{2\pi i m\delta'} e^{-2\pi i n\delta''} e^{-\pi i n\Omega n - 2\pi i n\zeta}\,\theta[\delta](\zeta|\Omega) \tag{A.4}$$

for $m, n \in \mathbb{Z}^h$. We note that from eq.(A.3) we have

$$|\theta[\zeta](0|\Omega)| = e^{-\pi \operatorname{Im}\zeta \frac{1}{\operatorname{Im}\Omega}\operatorname{Im}\zeta}\,|\theta(\zeta|\Omega)| \ . \tag{A.5}$$

It is immediate from the definition (A.3) that

$$\theta\begin{bmatrix}\delta'+m \\ \delta''+n\end{bmatrix}(\zeta|\Omega) = e^{2\pi i\delta' n}\theta\begin{bmatrix}\delta' \\ \delta''\end{bmatrix}(\zeta|\Omega) \tag{A.6}$$

[6] The mathematical background relevant for string perturbation theory is reviewed in Ref. [17]

for $m, n \in \mathbb{Z}^h$. Thus $\theta[\delta]$ only changes its phase if δ' and δ'' are shifted by integral vectors. The case in which $\delta', \delta'' \in (\mathbb{Z}/(2\mathbb{Z}))^h$ is important. In this situation, $[\delta] = \left[\begin{smallmatrix}\delta'\\ \delta''\end{smallmatrix}\right]$ is referred to as the spin structure, and we have

$$\theta[\delta](-\zeta|\Omega) = (-1)^{4\delta'\delta''}\theta[\delta](\zeta|\Omega) \ . \tag{A.7}$$

It follows that $\theta[\delta](\zeta|\Omega)$ is an even or odd function depending on whether $4\delta'\delta''$ is an even or odd integer. $[\delta]$ is accordingly referred to as the even spin structure or the odd spin structure.

Prime form

Let $[s] = \left[\begin{smallmatrix}s'\\ s''\end{smallmatrix}\right]$ be an odd spin structure. The prime form $E(z, w)$ is defined [18, 19] as

$$E(z, w) = \frac{\theta[s]\left(\int_w^z \omega \middle| \Omega\right)}{h_s(z)h_s(w)} \ , \tag{A.8}$$

where

$$h_s(z) = \sqrt{\sum_{j=1}^{h} \frac{\partial\theta[s]}{\partial\zeta_j}(0|\Omega)\omega_j(z)} \tag{A.9}$$

is a section of the spin bundle corresponding to $[s]$. The prime form $E(z, w)$ can be regarded as a $\left(-\frac{1}{2}, 0\right)$ form in each variable on the universal covering of Σ, whose transformation laws can be obtained from eq.(A.4) as follows: When z is moved around a_j cycle once, $E(z, w)$ is invariant up to a sign; whereas when z is moved around b_j cycle once, it transforms as

$$E(z, w) \mapsto \pm e^{-\pi i\Omega_{jj} - 2\pi i \int_w^z \omega_j} E(z, w) \ . \tag{A.10}$$

$E(z, w)$ satisfies $E(z, w) = -E(w, z)$, and for $z \sim w$ it behaves as

$$E(z, w) = (z - w) + \mathcal{O}\left((z - w)^3\right) \ . \tag{A.11}$$

Arakelov metric and Arakelov Green's function

Let us define $\mu_{z\bar{z}}$ as

$$\mu_{z\bar{z}} \equiv \frac{1}{2h}\omega(z)\frac{1}{\mathrm{Im}\,\Omega}\bar{\omega}(\bar{z}) \ . \tag{A.12}$$

We note that

$$\int_\Sigma dz \wedge d\bar{z}\, i\mu_{z\bar{z}} = 1 \ , \tag{A.13}$$

which follows from

$$\int_\Sigma \omega_j \wedge \bar{\omega}_k = -2i\,\mathrm{Im}\,\Omega_{jk} \ . \tag{A.14}$$

The Arakelov metric on Σ,

$$ds_{\mathrm{A}}^2 = 2g^{\mathrm{A}}_{z\bar{z}} dz d\bar{z} \ , \tag{A.15}$$

is defined so that its scalar curvature $R^{\mathrm{A}} \equiv -2g^{\mathrm{A}z\bar{z}}\partial\bar{\partial}\ln g^{\mathrm{A}}_{z\bar{z}}$ satisfies

$$g^{\mathrm{A}}_{z\bar{z}} R^{\mathrm{A}} = -8\pi(h-1)\mu_{z\bar{z}} \ . \tag{A.16}$$

This condition determines $g^{\mathrm{A}}_{z\bar{z}}$ only up to an overall constant, which we will choose later.

The Arakelov Green's function $G^{\mathrm{A}}(z,\bar{z};w,\bar{w})$ with respect to the Arakelov metric is defined to satisfy

$$\begin{aligned} &-\partial_z\partial_{\bar{z}} G^{\mathrm{A}}(z,\bar{z};w,\bar{w}) = 2\pi\delta^2(z-w) - 2\pi\mu_{z\bar{z}} \ , \\ &\int_\Sigma dz\wedge d\bar{z}\, i\mu_{z\bar{z}} G^{\mathrm{A}}(z,\bar{z};w,\bar{w}) = 0 \ . \end{aligned} \tag{A.17}$$

One can obtain a more explicit form of $G^{\mathrm{A}}(z,\bar{z};w,\bar{w})$ by solving eq.(A.17) for $G^{\mathrm{A}}(z,\bar{z};w,\bar{w})$. Using eq.(A.11), we have

$$\partial_z\partial_{\bar{z}}\ln F(z,\bar{z};w,\bar{w}) = -2\pi i\delta^2(z-w) - 2\pi h\mu_{z\bar{z}} \ , \tag{A.18}$$

where $F(z,\bar{z};w,\bar{w})$ is the $\left(-\frac{1}{2},-\frac{1}{2}\right)\times\left(-\frac{1}{2},-\frac{1}{2}\right)$ form on $\Sigma\times\Sigma$ defined as

$$F(z,\bar{z};w,\bar{w}) = \exp\left[-2\pi\,\mathrm{Im}\int_w^z \omega\frac{1}{\mathrm{Im}\,\Omega}\,\mathrm{Im}\int_w^z\omega\right]|E(z,w)|^2 \ . \tag{A.19}$$

Putting eqs.(A.18) and (A.16) together, we find that $G^{\mathrm{A}}(z,\bar{z};w,\bar{w})$ is given by

$$G^{\mathrm{A}}(z,\bar{z};w,\bar{w}) = -\ln F(z,\bar{z};w,\bar{w}) - \frac{1}{2}\ln\left(2g^{\mathrm{A}}_{z\bar{z}}\right) - \frac{1}{2}\ln\left(2g^{\mathrm{A}}_{w\bar{w}}\right) \ , \tag{A.20}$$

up to an additive constant independent of $z,\bar{z}$ and $w,\bar{w}$. This possible additive constant can be absorbed into the ambiguity in the overall constant of $g^{\mathrm{A}}_{z\bar{z}}$ mentioned above. It is required that eq.(A.20) holds exactly as it is [35, 21, 22]. This implies that

$$2g^{\mathrm{A}}_{z\bar{z}} = \lim_{w\to z}\exp\left[-G^{\mathrm{A}}(z,\bar{z};w,\bar{w}) - \ln|z-w|^2\right] \ , \tag{A.21}$$

and the overall constant of $g^{\mathrm{A}}_{z\bar{z}}$ is, in principle, determined by the second relation in eq.(A.17).

Mandelstam mapping

Here we illustrate several properties of the Mandelstam mapping (2.3).

The divisor $D_{d\rho} = \sum_{I=1}^{2h-2+N} z_I - \sum_{r=1}^{N} Z_r$ of the one-form $d\rho$ satisfies

$$\sum_{I=1}^{2h-2+N} \int_{P_0}^{z_I} \omega - \sum_{r=1}^{N} \int_{P_0}^{Z_r} \omega = 2\Delta \qquad (\mathrm{mod}\ \mathbb{Z}^h + \mathbb{Z}^h \Omega) \ . \tag{A.22}$$

Here Δ is the vector of Riemann constants for P_0, which is defined in $J(\Sigma)$. Its j-th component Δ_j is given by

$$\Delta_j = -\frac{\Omega_{jj}}{2} + \frac{1}{2} + \sum_{k \neq j} \oint_{a_k} \omega_k(P') \int_{P_0}^{P'} \omega_j \ . \tag{A.23}$$

From the singular behavior of the Mandelstam metric (2.5), one can find that it satisfies the differential equation

$$-\partial\bar{\partial} \ln |\partial\rho(z)|^2 = i2\pi \left(\sum_I \delta^2(z - z_I) - \sum_r \delta^2(z - Z_r) \right) \ . \tag{A.24}$$

This can be solved as

$$|\partial\rho(z)|^2 = 2 g^{\mathrm{A}}_{z\bar{z}} e^{\chi(z,\bar{z})} \ , \tag{A.25}$$

with

$$\chi(z,\bar{z}) = \sum_{r=1}^{N} G^{\mathrm{A}}(z; Z_r) - \sum_{I=1}^{2h-2+N} G^{\mathrm{A}}(z; z_I) + c \ , \tag{A.26}$$

where c is a constant independent of $z, \bar{z}$ but may depend on moduli. We here and henceforth suppress the anti-holomorphic coordinate dependence of G^{A} for brevity of notation. Looking at the behaviors of eq.(A.25) around $z \sim z_I$ and $z \sim Z_r$, one finds that

$$\begin{aligned} \left|\partial^2\rho(z_I)\right|^2 &= \left(2 g^{\mathrm{A}}_{z_I \bar{z}_I}\right)^2 \exp\left[-\sum_{J \neq I} G^{\mathrm{A}}(z_I; z_J) + \sum_r G^{\mathrm{A}}(z_I; Z_r) + c \right] \ , \\ |\alpha_r|^2 &= \exp\left[-\sum_I G^{\mathrm{A}}(z_I; Z_r) + \sum_{s \neq r} G^{\mathrm{A}}(Z_r; Z_s) + c \right] \ , \end{aligned} \tag{A.27}$$

and thus

$$\begin{aligned} \prod_I \left|\partial^2\rho(z_I)\right|^2 &= e^{(2h-2+N)c} \prod_I \left(2 g^{\mathrm{A}}_{z_I \bar{z}_I}\right)^2 \exp\left[-2\sum_{I<J} G^{\mathrm{A}}(z_I; z_J) + \sum_{I,r} G^{\mathrm{A}}(z_I; Z_r) \right] , \\ \prod_r |\alpha_r|^2 &= e^{Nc} \exp\left[-\sum_{I,r} G^{\mathrm{A}}(z_I; Z_r) + 2\sum_{r<s} G^{\mathrm{A}}(Z_r; Z_s) \right] . \end{aligned} \tag{A.28}$$

B Evaluation of Z^{LC}

In this appendix, we will evaluate the partition function $(Z^{\mathrm{LC}})^{\frac{d-2}{24}}$ for the transverse coordinates X^i. In the following, we consider the case $d = 26$ to get the partition function Z^{LC}.

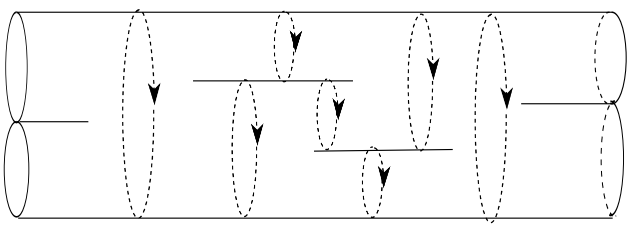

Figure 2: The contours $C_{\mathcal{I}}$.

Z^{LC} can be obtained by integrating the change $\delta \ln Z^{\mathrm{LC}}$ under the variation of moduli parameters, as in the tree case [11]. Since Z^{LC} is the partition function, if we vary the lengths and the twist angles of the internal propagators of the light-cone diagram, the change of Z^{LC} is given in terms of the expectation values of the Hamiltonian and the rotation generator as

$$\delta \ln Z^{\mathrm{LC}} = \sum_{\mathcal{I}} \delta \mathcal{T}_{\mathcal{I}} \oint_{C_{\mathcal{I}}} \frac{d\rho}{2\pi i} \left\langle T^{\mathrm{tr}}_{\rho\rho} \right\rangle^{X^i} + \mathrm{c.c.}\,. \tag{B.1}$$

Here $\mathcal{I}$ labels the internal lines of the light-cone diagram Σ and $C_{\mathcal{I}}$ denotes the contour going around it as depicted in Figure 2. $\mathcal{T}_{\mathcal{I}}$ is defined as

$$\mathcal{T}_{\mathcal{I}} = T_{\mathcal{I}} + i\alpha_{\mathcal{I}}\theta_{\mathcal{I}}\,, \tag{B.2}$$

where $T_{\mathcal{I}}$ denotes the length of the $\mathcal{I}$-th internal line and $\alpha_{\mathcal{I}}$, $\theta_{\mathcal{I}}$ denote the string-length and the twist angle for the propagator. $\mathrm{Re}\,\delta\mathcal{T}_{\mathcal{I}}$'s should satisfy some linear constraints so that the variation corresponds to that of the shape of a light-cone diagram. $\left\langle T^{\mathrm{tr}}_{\rho\rho} \right\rangle^{X^i}$ denotes the expectation value of the energy-momentum tensor $T^{\mathrm{tr}}_{\rho\rho}$ on the light-cone diagram for the worldsheet bosons X^i $(i = 1, \ldots, 24)$ corresponding to the transverse spacetime coordinates.

What we would like to do in the following is to calculate the right hand side of eq.(B.1) and integrate it. The variation we consider here corresponds to that of only a subset of $6h-6+2N$ moduli parameters. We do not consider the variation of $\alpha_{\mathcal{I}}$'s which are not fixed by the momentum conservation, namely that of the $+$ components of the loop momenta. Such variations correspond to integration cycles depicted in Figure 3. Therefore integrating the right hand side of eq.(B.1), integration constants depending on these parameters are left undetermined. We will fix these imposing the factorization conditions in subsection B.2.

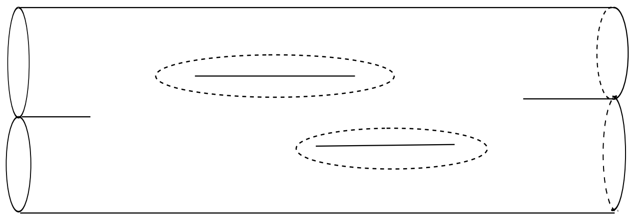

Figure 3: The cycles corresponding to the variations of the + components of the loop momenta.

B.1 Integration of the right hand side of eq.(B.1)

In order to integrate the right hand side of eq.(B.1), we introduce a convenient way to parametrize the moduli of the surface. As is depicted in Figure 4, cutting along h cycles with constant $\mathrm{Re}\,\rho$, one can make Σ into a surface with no handles but with $2h$ holes. By attaching $2h$ semi-infinite cylinders to the holes, it is possible to get a tree light-cone diagram, which is denoted by $\tilde{\Sigma}$. Let $\tilde{\rho}\,(z)$ be the Mandelstam mapping which maps $\mathbb{C}\cup\{\infty\}$ to $\tilde{\Sigma}$:

$$\begin{array}{rcl} \tilde{\rho}\ :\ \mathbb{C}\cup\{\infty\} & \longrightarrow & \tilde{\Sigma} \\ z & \mapsto & \tilde{\rho}\,(z)\ . \end{array} \tag{B.3}$$

$\tilde{\rho}\,(z)$ has the form

$$\tilde{\rho}\,(z)=\sum_{r=1}^{N}\alpha_r\ln(z-Z_r)+\sum_{A=1}^{h}\beta_A\ln\frac{z-Q_A}{z-R_A}\,. \tag{B.4}$$

Here β_A $(A=1,\cdots,h)$ are real positive parameters corresponding to the lengths of the h cycles along which the surface Σ is cut. The surface Σ can be obtained from $\tilde{\Sigma}$ by discarding the $2h$ semi-infinite cylinders and identifying the boundaries. Therefore we can use the z coordinate to describe Σ and we do so in the rest of this subsection. The Mandelstam mapping $\rho(z)$ can be given as

$$\rho(z)=\tilde{\rho}(z)+(\text{purely imaginary constant})\,, \tag{B.5}$$

and Σ corresponds to $\mathbb{C}\cup\{\infty\}$ with disks D_{Q_A}, D_{R_A} $(A=1,\cdots,h)$ around Q_A, R_A excised. We identify $z\in\partial D_{Q_A}$ and $w\in\partial D_{R_A}$ if

$$\tilde{\rho}\,(z)=\tilde{\rho}\,(w)+i\beta_A(\theta_A+2\pi n)\,, \tag{B.6}$$

for $n\in\mathbb{Z}$.

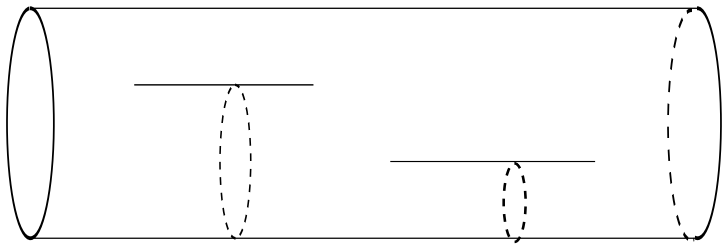

Figure 4: The h cycles along which we cut the light-cone diagram to make the Riemann surface Σ corresponding to the light-cone diagram into a surface with no handles but with $2h$ holes.

From the construction above, one can see that it is possible to associate the parameters $Z_r, \beta_A, \theta_A, Q_A, R_A$ with any light-cone diagram. Therefore, with α_r $(= 2p_r^+)$ fixed, the shape of Σ is parametrized locally by $Z_r, \beta_A, \theta_A, Q_A, R_A$ modded out by the 6 conformal transformations on $\mathbb{C} \cup \{\infty\}$. Thus we have $2N + h + h + 2h + 2h - 6 = 6h - 6 + 2N$ real parameters, the number of which coincides with that of the moduli parameters of the punctured Riemann surface Σ. A variation of the complex structure of Σ corresponds to a variation of these parameters. β_A's correspond to the loop momenta and the variation we consider here corresponds to the one with $\delta\beta_A = 0$. Under such a variation of the parameters, the rule of identification (B.6) is also changed as

$$(\tilde{\rho} + \delta\tilde{\rho})\,(z) = (\tilde{\rho} + \delta\tilde{\rho})\,(w + \delta w) + i\beta_A(\theta_A + \delta\theta_A + 2\pi n)\,. \tag{B.7}$$

Accordingly we obtain

$$\delta\tilde{\rho}\,(z) - \delta\tilde{\rho}\,(w) = \delta w \partial\tilde{\rho}\,(w) + i\beta_A \delta\theta_A\,. \tag{B.8}$$

Therefore as a function on Σ, the variation $\delta\rho\,(z)$ is discontinuous along the cycle corresponding to ∂D_{Q_A}.

In terms of the quantities defined using the z coordinate, the right hand side of eq.(B.1) can be expressed as

$$\delta \ln Z^{\mathrm{LC}} = \sum_{\mathcal{I}} \delta\mathcal{T}_{\mathcal{I}} \oint_{C_{\mathcal{I}}} \frac{dz}{2\pi i} \frac{1}{\partial\rho(z)} \left(\left\langle T_{zz}^{\mathrm{tr}} \right\rangle^{X^i} - 2\{\rho, z\} \right) + \text{c.c.}\,. \tag{B.9}$$

Here $\{\rho, z\}$ denotes the Schwarzian derivative, which is given by

$$\begin{aligned} -2\,\{\rho, z\} &= -2\frac{\partial^3\rho}{\partial\rho} + 3\left(\frac{\partial^2\rho}{\partial\rho}\right)^2 \\ &= \left(\partial \ln |\partial\rho|^2\right)^2 - 2\partial^2 \ln |\partial\rho|^2\,. \end{aligned} \tag{B.10}$$

Calculation of the right hand side of eq.(B.9)

$\mathcal{T}_\mathcal{I}$ can be expressed as

$$\mathcal{T}_\mathcal{I}=\rho\left(z_{\mathcal{I}+}\right)-\rho\left(z_{\mathcal{I}-}\right)\ , \tag{B.11}$$

where $z_{\mathcal{I}+}$ and $z_{\mathcal{I}-}$ are the z coordinates of the interaction points on the two sides of the $\mathcal{I}$-th internal line. Rewriting each term on the right hand side of eq.(B.9) as

$$\begin{aligned}
&\delta\mathcal{T}_\mathcal{I}\oint_{C_\mathcal{I}}\frac{dz}{2\pi i}\frac{1}{\partial\rho\left(z\right)}\left(\left\langle T^{\mathrm{tr}}_{zz}\right\rangle^{X^i}-2\left\{\rho,z\right\}\right)\\
&\qquad=\oint_{C_\mathcal{I}}\frac{dz}{2\pi i}\frac{\delta\rho\left(z\right)-\delta\rho\left(z_{\mathcal{I}-}\right)}{\partial\rho\left(z\right)}\left(\left\langle T^{\mathrm{tr}}_{zz}\right\rangle^{X^i}-2\left\{\rho,z\right\}\right)\\
&\qquad\quad-\oint_{C_\mathcal{I}}\frac{dz}{2\pi i}\frac{\delta\rho\left(z\right)-\delta\rho\left(z_{\mathcal{I}+}\right)}{\partial\rho\left(z\right)}\left(\left\langle T^{\mathrm{tr}}_{zz}\right\rangle^{X^i}-2\left\{\rho,z\right\}\right)\ ,
\end{aligned} \tag{B.12}$$

and deforming the contours, we obtain

$$\begin{aligned}
\delta\ln Z^{\mathrm{LC}}\ =\ &-\sum_r\oint_{Z_r}\frac{dz}{2\pi i}\frac{\delta\rho\left(z\right)-\delta\rho\left(z_{I(r)}\right)}{\partial\rho\left(z\right)}\left(\left\langle T^{\mathrm{tr}}_{zz}\right\rangle^{X^i}-2\left\{\rho,z\right\}\right)\\
&-\sum_I\oint_{z_I}\frac{dz}{2\pi i}\frac{\delta\rho\left(z\right)-\delta\rho\left(z_I\right)}{\partial\rho\left(z\right)}\left(\left\langle T^{\mathrm{tr}}_{zz}\right\rangle^{X^i}-2\left\{\rho,z\right\}\right)\\
&-\sum_A\oint_{\partial D_{Q_A}}\frac{dz}{2\pi i}\frac{\delta\rho\left(z\right)}{\partial\rho\left(z\right)}\left(\left\langle T^{\mathrm{tr}}_{zz}\right\rangle^{X^i}-2\left\{\rho,z\right\}\right)\\
&-\sum_A\oint_{\partial D_{R_A}}\frac{dz}{2\pi i}\frac{\delta\rho\left(z\right)}{\partial\rho\left(z\right)}\left(\left\langle T^{\mathrm{tr}}_{zz}\right\rangle^{X^i}-2\left\{\rho,z\right\}\right)\\
&+\mathrm{c.c.}\ .
\end{aligned} \tag{B.13}$$

The third and the fourth terms do not cancel with each other because of the discontinuity of $\delta\rho\left(z\right)$ mentioned above.

While $\langle T^{\mathrm{tr}}_{zz}\rangle^{X^i}$ is regular for $z\sim Z_r$ and z_I, the Schwarzian derivative $-2\{\rho,z\}$ given in eq.(B.10) behaves as

$$\begin{aligned}
-2\left\{\rho,z\right\}\ \sim\ &\frac{-1}{\left(z-Z_r\right)^2}\\
&+\frac{1}{z-Z_r}\frac{\partial}{\partial Z_r}\left(2\sum_I G^{\mathrm{A}}\left(Z_r;z_I\right)-2\sum_{s\neq r}G^{\mathrm{A}}\left(Z_r;Z_s\right)\ -\ln g^{\mathrm{A}}_{Z_r\bar{Z}_r}\right)
\end{aligned} \tag{B.14}$$

for $z\sim Z_r$ and

$$\begin{aligned}
-2\left\{\rho,z\right\}\ \sim\ &\frac{3}{\left(z-z_I\right)^2}\\
&+\frac{1}{z-z_I}\frac{\partial}{\partial z_I}\left(-2\sum_{J\neq I}G^{\mathrm{A}}\left(z_I;z_J\right)+2\sum_r G^{\mathrm{A}}\left(z_I;Z_r\right)\ +3\ln g^{\mathrm{A}}_{z_I\bar{z}_I}\right)
\end{aligned} \tag{B.15}$$

for $z \sim z_I$, as can be derived using eq.(A.25).

We here consider a variation of the form $Z_r \to Z_r+\delta Z_r$, $Q_A \to Q_A+\delta Q_A$, $R_A \to R_A+\delta R_A$ and

$$\begin{aligned}
\frac{\delta\rho(z)-\delta\rho(z_{I^{(r)}})}{\partial\rho(z)} &\sim -\delta Z_r-(z-Z_r)\,\delta\bar{N}_{00}^{rr}+\mathcal{O}\left((z-Z_r)^2\right) && \text{for } z\sim Z_r\ , \\
\frac{\delta\rho(z)-\delta\rho(z_I)}{\partial\rho(z)} &\sim -\delta z_I+(z-z_I)\,\frac{1}{2}\delta\left(\ln\partial^2\rho(z_I)\right)+\mathcal{O}\left((z-z_I)^2\right) && \text{for } z\sim z_I\ ,
\end{aligned} \tag{B.16}$$

where

$$\delta z_I = z_I(Z_r+\delta Z_r, Q_A+\delta Q_A, R_A+\delta R_A)-z_I(Z_r,Q_A,R_A)\,. \tag{B.17}$$

$\bar{N}_{00}^{rr}$ denotes one of the Neumann coefficients and is given by

$$\begin{aligned}
\bar{N}_{00}^{rr} &\equiv \lim_{z\to Z_r}\left[\frac{\rho(z_{I^{(r)}})-\rho(z)}{\alpha_r}+\ln(z-Z_r)\right] \\
&= \frac{\rho(z_{I^{(r)}})}{\alpha_r}-\sum_{s\neq r}\frac{\alpha_s}{\alpha_r}\ln E(Z_r,Z_s)+\frac{2\pi i}{\alpha_r}\int_{P_0}^{Z_r}\omega\frac{1}{\operatorname{Im}\Omega}\sum_{s=1}^{N}\alpha_s\operatorname{Im}\int_{P_0}^{Z_s}\omega\ .
\end{aligned} \tag{B.18}$$

Using eqs.(B.14), (B.15) and (B.16), we can easily evaluate the first two terms on the right hand side of (B.13) and obtain

$$\begin{aligned}
\delta\ln Z^{\mathrm{LC}} &= \delta\left(-\sum_r\bar{N}_{00}^{rr}-\sum_I\frac{3}{2}\ln\partial^2\rho(z_I)\right) \\
&\quad+\sum_r\delta Z_r\frac{\partial}{\partial Z_r}\left(2\sum_I G^{\mathrm{A}}(Z_r,z_I)-2\sum_{s\neq r}G^{\mathrm{A}}(Z_r,Z_s)-\ln g^{\mathrm{A}}_{Z_r\bar{Z}_r}\right) \\
&\quad+\sum_I\delta z_I\frac{\partial}{\partial z_I}\left(-2\sum_{J\neq I}G^{\mathrm{A}}(z_I,z_J)+2\sum_r G^{\mathrm{A}}(z_I,Z_r)+3\ln g^{\mathrm{A}}_{z_I\bar{z}_I}\right) \\
&\quad-\sum_A\oint_{\partial D_{Q_A}}\frac{dz}{2\pi i}\frac{\delta\rho(z)}{\partial\rho(z)}\left(\left\langle T^{\mathrm{tr}}_{zz}\right\rangle^{X^i}-2\{\rho,z\}\right) \\
&\quad-\sum_A\oint_{\partial D_{R_A}}\frac{dz}{2\pi i}\frac{\delta\rho(z)}{\partial\rho(z)}\left(\left\langle T^{\mathrm{tr}}_{zz}\right\rangle^{X^i}-2\{\rho,z\}\right) \\
&\quad+\text{c.c.}\ .
\end{aligned} \tag{B.19}$$

In the following, we will show that there exists Z which satisfies

$$\delta\ln Z = \sum_r\delta Z_r\frac{\partial}{\partial Z_r}\left(2\sum_I G^{\mathrm{A}}(Z_r;z_I)-2\sum_{s\neq r}G^{\mathrm{A}}(Z_r;Z_s)-\ln g^{\mathrm{A}}_{Z_r\bar{Z}_r}\right)$$

$$
\begin{aligned}
&+\sum_I \delta z_I \frac{\partial}{\partial z_I}\left(-2\sum_{J\neq I} G^{\mathrm{A}}\left(z_I;z_J\right)+2\sum_r G^{\mathrm{A}}\left(z_I;Z_r\right)+3\ln g^{\mathrm{A}}_{z_I\bar{z}_I}\right) \\
&-\sum_A \oint_{\partial D_{Q_A}} \frac{dz}{2\pi i}\frac{\delta\rho\left(z\right)}{\partial\rho\left(z\right)}\left(\left\langle T^{\mathrm{tr}}_{zz}\right\rangle^{X^i}-2\left\{\rho,z\right\}\right) \\
&-\sum_A \oint_{\partial D_{R_A}} \frac{dz}{2\pi i}\frac{\delta\rho\left(z\right)}{\partial\rho\left(z\right)}\left(\left\langle T^{\mathrm{tr}}_{zz}\right\rangle^{X^i}-2\left\{\rho,z\right\}\right) \\
&+\mathrm{c.c.}\,,
\end{aligned}
\tag{B.20}
$$

under a variation of the parameters Z_r, Q_A, R_A. Then we get

$$
\delta\ln Z^{\mathrm{LC}}=\delta\left(-\sum_r 2\,\mathrm{Re}\,\bar{N}^{rr}_{00}-\sum_I\frac{3}{2}\ln\left|\partial^2\rho(z_I)\right|^2+\ln Z\right)\,. \tag{B.21}
$$

Correlation function Z

Let us consider a metric $ds^2=2g_{z\bar{z}}dzd\bar{z}$ on the Riemann surface Σ and define

$$
Z \;\equiv\; Z^X[g_{z\bar{z}}]^{23}\int_\Sigma [d\Phi]_{g_{z\bar{z}}}\, e^{-S[\Phi]}\delta\left(\Phi\left(z_0,\bar{z}_0\right)\right)\prod_{I=1}^{2h-2+N}\mathcal{O}_I\prod_{r=1}^{N}V_r\,. \tag{B.22}
$$

Here $S\left[\Phi\right]$ is the action for the boson Φ given as

$$
S\left[\Phi\right]\equiv\frac{1}{8\pi}\int d^2z\sqrt{g}\left[g^{ab}\partial_a\Phi\partial_b\Phi-2\sqrt{2}iR\Phi\right]\,, \tag{B.23}
$$

and $Z^X\left[g_{z\bar{z}}\right]$ is given in eq.(2.15). $\mathcal{O}_I$ and V_r are the vertex operators defined as

$$
\begin{aligned}
\mathcal{O}_I &\equiv \left(2g_{z_I\bar{z}_I}\right)^2 e^{i\sqrt{2}\Phi}\left(z_I,\bar{z}_I\right)\,, \\
V_r &\equiv \left(2g_{Z_r\bar{Z}_r}\right)^{-2} e^{-i\sqrt{2}\Phi}\left(Z_r,\bar{Z}_r\right)\,.
\end{aligned}
\tag{B.24}
$$

The operators $e^{i\sqrt{2}\Phi}\left(z_I,\bar{z}_I\right)$, $e^{-i\sqrt{2}\Phi}\left(Z_r,\bar{Z}_r\right)$ on the right hand side are normal ordered and $\mathcal{O}_I, V_r$ are defined to be Weyl invariant. $\delta\left(\Phi\left(z_0,\bar{z}_0\right)\right)$ is necessary to soak up the zero mode of Φ and Z does not depend on $z_0,\bar{z}_0$. The energy-momentum tensor for Φ is given as

$$
-\frac{1}{2}\partial\Phi\partial\Phi-\sqrt{2}i\left(\partial-\partial\ln g_{z\bar{z}}\right)\partial\Phi\,, \tag{B.25}
$$

and the Virasoro central charge is -23. Since $\mathcal{O}_I, V_r$ are Weyl invariant, Z does not depend on the metric $g_{z\bar{z}}$ and is a function of the moduli parameters $Z_r, \beta_A, \theta_A, Q_A, R_A$. With the metric $g_{z\bar{z}}$ taken to be the Arakelov metric $g^{\mathrm{A}}_{z\bar{z}}$, we can evaluate Z to be

$$
\begin{aligned}
Z \;\propto\; & Z^X[g^{\mathrm{A}}_{zz}]^{24}\prod_I\left(2g^{\mathrm{A}}_{z_I\bar{z}_I}\right)^3\prod_r\left(2g^{\mathrm{A}}_{Z_r\bar{Z}_r}\right)^{-1} \\
&\times\exp\left[-2\sum_{I<J}G^{\mathrm{A}}\left(z_I,z_J\right)-2\sum_{r<s}G^{\mathrm{A}}\left(Z_r;Z_s\right)+2\sum_{I,r}G^{\mathrm{A}}\left(z_I;Z_r\right)\right]\,,
\end{aligned}
\tag{B.26}
$$

using the Arakelov Green's function $G^{\mathrm{A}}(z;w)$ with respect to the Arakelov metric. We note that unlike the scalar field used in bosonization, Φ is not circle valued. Therefore contributions from the soliton sector are not included in eq.(B.26).

In the following, we would like to prove that Z thus defined satisfies eq.(B.20). Let us first rewrite Z in a factorized form. As we did earlier, cutting the light-cone diagram Σ and attaching semi-infinite cylinders, one can get a tree light-cone diagram $\tilde{\Sigma}$. We replace the cut propagators by

$$\sum_n e^{-Th_n}\left|n\right\rangle\left\langle n\right| = \sum_n e^{-Th_n}\mathcal{O}_n(0,0)\left|0\right\rangle\left\langle 0\right| I\circ\mathcal{O}_n(0,0)\,, \tag{B.27}$$

where $\{|n\rangle\}$ is a complete basis of the states, h_n denotes the weight of $|n\rangle$, $\mathcal{O}_n(z,\bar{z})$ is the local operator corresponding to the state $|n\rangle$ and $I(z)\equiv 1/z$ is the inversion. We denote by $f\circ\mathcal{O}(z,\bar{z})$ the transform of the local operator $\mathcal{O}(z,\bar{z})$ under the conformal transformation $(z,\bar{z})\to(f(z),\bar{f}(\bar{z}))$. One can express Z in terms of a correlation function on $\mathbb{C}\cup\{\infty\}$ as

$$\begin{aligned} Z \;=\; & \int_{\mathbb{C}\cup\{\infty\}}\left[dX^1\cdots dX^{23}d\Phi\right]e^{-S[\Phi]-S^X}\,\delta\left(\Phi\left(z_0,\bar{z}_0\right)\right)\prod_I\mathcal{O}_I\prod_r V_r \\ & \times\prod_A\left(\sum_n f_{Q_A}^{-1}\circ\mathcal{O}_n\left(0,0\right)f_{R_A}^{-1}\circ\mathcal{O}_n\left(0,0\right)\right)\,, \end{aligned} \tag{B.28}$$

where S^X denotes the worldsheet action for the free bosons $X^1,\ldots,X^{23}$, and

$$f_{Q_A}\left(z\right)=e^{\frac{1}{\beta_A}\tilde{\rho}(z)}\,,\qquad f_{R_A}\left(z\right)=e^{-\frac{1}{\beta_A}\tilde{\rho}(z)}\,. \tag{B.29}$$

Since the total central charge of the system vanishes, we do not have to specify the metric on $\mathbb{C}\cup\{\infty\}$ in eq.(B.28).

For any $f\left(z\right)$ regular at $z=0$, there exists an operator U_f of the form [36, 37, 38, 39]

$$U_f=e^{\oint\frac{dz}{2\pi i}v(z)T_{zz}+\mathrm{c.c.}}\,, \tag{B.30}$$

such that

$$U_f\mathcal{O}\left(0,0\right)\left|0\right\rangle = f\circ\mathcal{O}\left(0,0\right)\left|0\right\rangle\,. \tag{B.31}$$

The relation between $v\left(z\right)$ and $f\left(z\right)$ is given by

$$e^{v(z)\partial_z}z=f\left(z\right)\,, \tag{B.32}$$

and one can get $v(z)$ from $f(z)$ solving eq.(B.32). Suppose $f\to f+\delta f$ is an infinitesimal variation of f. From

$$\left(1+\frac{\delta f}{\partial f}\partial_z\right)e^{v(z)\partial_z}z=f\left(z\right)+\delta f(z)\,, \tag{B.33}$$

one can prove

$$U_f\left(1+\left(\oint \frac{dz}{2\pi i}\frac{\delta f}{\partial f}T_{zz}+\text{c.c.}\right)\right)=U_{f+\delta f}\,. \tag{B.34}$$

Using eq.(B.34), $\delta\ln Z$ under the variation of Z_r, Q_A, R_A is given as

$$\begin{aligned}
\delta\ln Z \;=\; & \frac{1}{Z}\int_{\mathbb{C}\cup\{\infty\}}\left[dX^1\cdots dX^{23}d\Phi\right]e^{-S[\Phi]-S^X}\delta\left(\Phi\left(z_0,\bar z_0\right)\right)\\
&\times\Bigg[\sum_I \delta z_I\partial\mathcal{O}_I\prod_{J\neq I}\mathcal{O}_J\prod_r V_r\prod_A\left(\sum_n f_{Q_A}^{-1}\circ\mathcal{O}_n\left(0,0\right)f_{R_A}^{-1}\circ\mathcal{O}_n\left(0,0\right)\right)\\
&\quad+\sum_r\delta Z_r\partial V_r\prod_I\mathcal{O}_I\prod_{s\neq r}V_s\prod_A\left(\sum_n f_{Q_A}^{-1}\circ\mathcal{O}_n\left(0,0\right)f_{R_A}^{-1}\circ\mathcal{O}_n\left(0,0\right)\right)\\
&\quad+\prod_I\mathcal{O}_I\prod_r V_r\sum_A\left(-\oint_{Q_A}\frac{dz}{2\pi i}\frac{\delta\tilde\rho}{\partial\tilde\rho}T_{zz}-\oint_{R_A}\frac{dz}{2\pi i}\frac{\delta\tilde\rho}{\partial\tilde\rho}T_{zz}\right)\\
&\qquad\times\prod_B\left(\sum_n f_{Q_B}^{-1}\circ\mathcal{O}_n\left(0,0\right)f_{R_B}^{-1}\circ\mathcal{O}_n\left(0,0\right)\right)\\
&\quad+\text{c.c.}\Bigg]\,.
\end{aligned}\tag{B.35}$$

The right hand side of eq.(B.35) can be written in terms of the correlation functions on Σ:

$$\begin{aligned}
\delta\ln Z \;=\; & \frac{1}{Z}\int_{\Sigma}\left[dX^1\cdots dX^{23}d\Phi\right]e^{-S[\Phi]-S^X}\delta\left(\Phi\left(z_0,\bar z_0\right)\right)\\
&\times\Bigg[\sum_I \delta z_I\partial\mathcal{O}_I\prod_{J\neq I}\mathcal{O}_J\prod_r V_r+\sum_r\delta Z_r\partial V_r\prod_I\mathcal{O}_I\prod_{s\neq r}V_s\\
&\quad+\prod_I\mathcal{O}_I\prod_r V_r\sum_A\left(-\oint_{Q_A}\frac{dz}{2\pi i}\frac{\delta\rho}{\partial\rho}T_{zz}-\oint_{R_A}\frac{dz}{2\pi i}\frac{\delta\rho}{\partial\rho}T_{zz}\right)\\
&\quad+\text{c.c.}\Bigg]\,.
\end{aligned}\tag{B.36}$$

It is straightforward to evaluate the contribution of the first two terms of the parenthesis and we get the first two terms on the right hand side of eq.(B.20). The energy-momentum tensor T_{zz} is given as

$$T_{zz}=-\frac{1}{2}\left(\partial\Phi\right)^2-\sqrt{2}i\left(\partial-\partial\ln g^{\mathrm{A}}_{z\bar z}\right)\partial\Phi-\sum_{i=1}^{23}\frac{1}{2}\left(\partial X^i\right)^2+\left(\partial\ln g^{\mathrm{A}}_{z\bar z}\right)^2-2\partial^2\ln g^{\mathrm{A}}_{z\bar z}\ , \tag{B.37}$$

if we take the metric on Σ to be the Arakelov metric. The expectation value of T_{zz} can be calculated to be

$$
\begin{aligned}
\langle T_{zz} \rangle &\equiv \frac{1}{Z}\int_{\Sigma}\left[dX^{1}\cdots dX^{23}d\Phi\right]e^{-S[\Phi]-S^{X}}\,T_{zz}\,\delta\left(\Phi(z_0,\bar{z}_0)\right)\prod_{I}\mathcal{O}_{I}\prod_{r}V_{r} \\
&= -\frac{1}{2}\left(\partial\Phi_{\mathrm{cl}}\right)^{2}-\sqrt{2}i\left(\partial-\partial\ln g^{\mathrm{A}}_{z\bar{z}}\right)\partial\Phi_{\mathrm{cl}} \\
&\quad +24\lim_{w\to z}\left(-\frac{1}{2}\partial_{z}\partial_{w}G^{\mathrm{A}}\left(z;w\right)-\frac{\frac{1}{2}}{\left(z-w\right)^{2}}\right)+\left(\partial\ln g^{\mathrm{A}}_{z\bar{z}}\right)^{2}-2\partial^{2}\ln g^{\mathrm{A}}_{z\bar{z}} \\
&= \left\langle T^{\mathrm{tr}}_{zz}\right\rangle^{X^{i}}-2\left\{\rho,z\right\}\,, \qquad\qquad (\mathrm{B.38})
\end{aligned}
$$

where

$$
\Phi_{\mathrm{cl}}\left(z,\bar{z}\right)=i\sqrt{2}\sum_{I}G^{\mathrm{A}}\left(z;z_{I}\right)-i\sqrt{2}\sum_{r}G^{\mathrm{A}}\left(z;Z_{r}\right)+c'\,, \qquad (\mathrm{B.39})
$$

and c' is a constant which is fixed by the condition $\Phi_{\mathrm{cl}}\left(z_0,\bar{z}_0\right)=0$. Substituting eq.(B.38) into eq.(B.36), we get eq.(B.20) and thus eq.(B.21).

Partition function Z^{LC}

Substituting eq.(B.26) into eq.(B.21), we eventually obtain

$$
Z^{\mathrm{LC}}=Z^{X}[g^{\mathrm{A}}_{z\bar{z}}]^{24}e^{-\Gamma\left[g^{\mathrm{A}}_{z\bar{z}},\,\ln|\partial\rho|^{2}\right]}\,, \qquad (\mathrm{B.40})
$$

where

$$
\begin{aligned}
e^{-\Gamma\left[g^{\mathrm{A}}_{z\bar{z}},\,\ln|\partial\rho|^{2}\right]} &= \mathcal{C}(\beta_{A})\prod_{r}\left[e^{-2\mathrm{Re}\bar{N}^{rr}_{00}}\left(2g^{\mathrm{A}}_{Z_{r}\bar{Z}_{r}}\right)^{-1}\right]\prod_{I}\left[\left|\partial^{2}\rho\left(z_{I}\right)\right|^{-3}\left(2g^{\mathrm{A}}_{z_{I}\bar{z}_{I}}\right)^{3}\right] \\
&\quad\times\exp\left[-2\sum_{I<J}G^{\mathrm{A}}\left(z_{I};z_{J}\right)-2\sum_{r<s}G^{\mathrm{A}}\left(Z_{r};Z_{s}\right)+2\sum_{I,r}G^{\mathrm{A}}\left(z_{I};Z_{r}\right)\right]. \qquad (\mathrm{B.41})
\end{aligned}
$$

Here $\mathcal{C}(\beta_{A})$ is an integration constant independent of the parameters Z_{r},Q_{A},R_{A}. Using eq.(A.28), one can express $e^{-\Gamma\left[g^{\mathrm{A}}_{z\bar{z}},\,\ln|\partial\rho|^{2}\right]}$ given in eq.(B.41) as

$$
e^{-\Gamma\left[g^{\mathrm{A}}_{z\bar{z}},\,\ln|\partial\rho|^{2}\right]}=\mathcal{C}(\beta_{A})e^{-2(h-1)c}\prod_{r}\left[e^{-2\,\mathrm{Re}\,\bar{N}^{rr}_{00}}(2g^{\mathrm{A}}_{Z_{r}\bar{Z}_{r}})^{-1}\alpha_{r}^{-2}\right]\prod_{I}\left[\left|\partial^{2}\rho(z_{I})\right|^{-1}2g^{\mathrm{A}}_{z_{I}\bar{z}_{I}}\right]\,. \qquad (\mathrm{B.42})
$$

Applying the bosonization technique [40, 41, 35, 21, 22] to the bc system in which the weight of the b-ghost is 1 and that of the c-ghost is 0, one can have the following expression of $Z^{X}[g^{\mathrm{A}}_{z\bar{z}}]$,

$$
Z^{X}[g^{\mathrm{A}}_{z\bar{z}}]^{24}\propto e^{2\delta(\Sigma)}\,, \qquad (\mathrm{B.43})
$$

in terms of the Faltings' invariant $\delta(\Sigma)$ [42] defined by

$$e^{-\frac{1}{4}\delta(\Sigma)} = (\det \operatorname{Im}\Omega)^{\frac{3}{2}} |\theta[\xi](0|\Omega)|^2 \frac{\prod_{i=1}^{h}\left(2g^{\mathrm{A}}_{\hat{z}_i\bar{\hat{z}}_i}\right)}{|\det \omega_j(\hat{z}_i)|^2} \times \exp\left[-\sum_{i<j} G^{\mathrm{A}}(\hat{z}_i;\hat{z}_j) + \sum_i G^{\mathrm{A}}(\hat{z}_i;\hat{w})\right]. \quad \text{(B.44)}$$

Here $\hat{z}_i$ $(i = 1, \ldots, h)$ and $\hat{w}$ are arbitrary points on Σ, and $\xi \in J(\Sigma)$ is defined as

$$\xi \equiv \sum_{i=1}^{h} \int_{P_0}^{\hat{z}_i} \omega - \int_{P_0}^{\hat{w}} \omega - \Delta\ . \quad \text{(B.45)}$$

Putting eqs.(B.42) and (B.43) together, we obtain the following expression of Z^{LC},

$$Z^{\mathrm{LC}} = \mathcal{C}_{h,N}(\beta_A) e^{-2(h-1)c} e^{2\delta(\Sigma)} \prod_r \left[e^{-2\operatorname{Re}\bar{N}^{rr}_{00}} (2g^{\mathrm{A}}_{Z_r\bar{Z}_r})^{-1} \alpha_r^{-2}\right] \prod_I \left[\left|\partial^2\rho(z_I)\right|^{-1} 2g^{\mathrm{A}}_{z_I\bar{z}_I}\right]. \quad \text{(B.46)}$$

The factor $\mathcal{C}_{h,N}(\beta_A)$ is left undetermined. Since the expression (B.46) is given in terms of the quantities which is independent of the choice of the local complex coordinate z, eq.(B.46) is valid for the coordinates other than the z coordinate used in this subsection to derive it.

B.2 Factorization

We can fix the $\mathcal{C}_{h,N}(\beta_A)$ using the factorization condition. By varying lengths of the propagators, it is possible to realize the degeneration limit of the Riemann surface Σ. Taking such a limit of Z^{LC} in eq.(B.46) and imposing the factorization conditions, we are able to get relations which $\mathcal{C}_{h,N}(\beta_A)$'s with various h, N should satisfy. In the following, we first consider the degeneration of an h-loop diagram with $h > 1$ depicted in Figure 5 and show that $\mathcal{C}_{h,N}(\beta_A)$ with $h > 1$ can be expressed by $\mathcal{C}_{1,N}(\beta_A)$. We then consider the degeneration depicted in Figure 7, in which a one-loop diagram is separated into two tree diagrams. The partition functions for the tree diagrams are given in Ref. [11] and we are able to get $\mathcal{C}_{1,N}(\beta_A)$.

B.2.1 $h > 1$ case

In the degeneration depicted in Figure 5, a zero homology cycle is pinched to a node p and the worldsheet Riemann surface Σ will be separated into two disconnected components Σ_1 and Σ_2 in Figure 5 (b). We denote the node p by p_1 or p_2 depending on whether it is regarded as a puncture in Σ_1 or that in Σ_2. Let Σ_1, Σ_2 be of genus h_1, h_2 and with $N_1 + 1$, $N_2 + 1$

punctures respectively, where $h_1 + h_2 = h$ and $N_1 + N_2 = N$. Since the partition functions on Σ_1 and Σ_2 are again expressed as eq.(B.46), we can obtain a relation between $\mathcal{C}_{h,N}(\beta)$ and $\mathcal{C}_{h_1,N_1}(\beta_{A_1})\mathcal{C}_{h_2,N_2}(\beta_{A_2})$ by examining such degenerations.

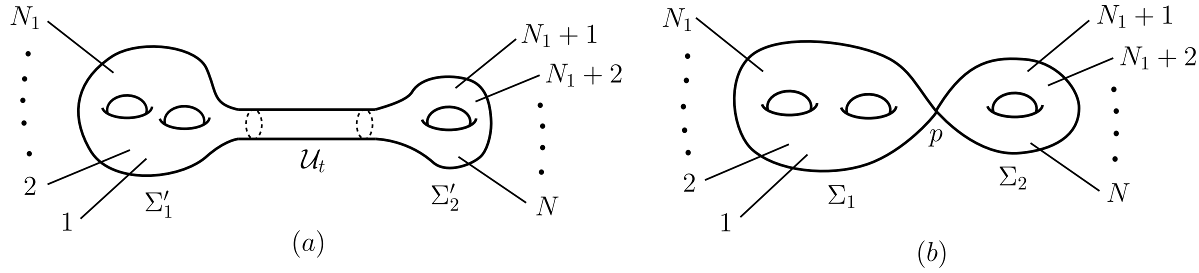

Figure 5: The degeneration of Riemann surface Σ. (a) The degeneration process described by the plumbing fixture $\mathcal{U}_t$. (b) A cycle homologous to zero is pinched to the point p.

The degeneration that we consider here can be described as the process in which the plumbing fixture $\mathcal{U}_t$ parametrized by a complex parameter t with $|t| < 1$ disconnects the Riemann surface $\Sigma(t)$ into Σ_1 and Σ_2 as $t \to 0$. In Figure 5 (a), we denote by Σ'_1 and Σ'_2 the components of the complement of $\mathcal{U}_t$ in $\Sigma(t)$. In this process, the basis of the homology cycles $\{a_j, b_j\}$ $(j = 1, \ldots, h)$ will be divided into the homology basis on Σ_1 and that on Σ_2. We assume that $\{a_j, b_j\}$ are ordered so that $\{a_{j_1}, b_{j_1}\}$ $(1 \leq j_1 \leq h_1)$ are cycles in Σ'_1 and $\{a_{j_2}, b_{j_2}\}$ $(h_1 + 1 \leq j_2 \leq h)$ are those in Σ'_2. Similarly, the set of the punctures Z_r $(r = 1, \ldots, N)$ will be separated as $Z_r = (Z_{r_1}, Z_{r_2})$ $(1 \leq r_1 \leq N_1,\ N_1 + 1 \leq r_2 \leq N)$ with $Z_{r_1} \in \Sigma'_1$ and $Z_{r_2} \in \Sigma'_2$, and the set of the interaction points z_I $(I = 1, \ldots, 2h - 2 + N)$ as $z_I = (z_{I_1}, z_{I_2})$ $(1 \leq I_1 \leq 2h_1 - 1 + N_1,\ 2h_1 + N_1 \leq I_2 \leq 2h - 2 + N)$ with $z_{I_1} \in \Sigma'_1$ and $z_{I_2} \in \Sigma'_2$. Without the loss of generality, we assume that the base point P_0 of the Abel-Jacobi map on Σ lies in Σ'_1.

Asymptotics of Arakelov Green's function and Arakelov metric

The canonical basis of the holomorphic one-forms $\omega_j(z;t)$ $(j = 1, \ldots, h)$ of $\Sigma(t)$ tends to the combined bases of holomorphic one-forms $\omega^{(1)}_{j_1}(z)$ $(1 \leq j_1 \leq h_1)$ and $\omega^{(2)}_{j_2}(z)$ $(h_1+1 \leq j_2 \leq h)$ of Σ_1 and Σ_2 as [18, 43, 44]

$$\omega_{j_1}(z;t) = \begin{cases} \omega^{(1)}_{j_1}(z) + \mathcal{O}(t^2) & \text{for } z \in \Sigma'_1 \\ -t\omega^{(1)}_{j_1}(p_1)\omega^{(2)}(z, p_2) + \mathcal{O}(t^2) & \text{for } z \in \Sigma'_2 \end{cases}, \tag{B.47}$$

and similarly for $\omega_{j_2}(z;t)$ with the roles of Σ_1 and Σ_2 interchanged. Here $\omega^{(1)}(z,w)$ denotes the abelian differential of the second kind on Σ_1 defined as

$$\omega^{(1)}(z,w) = dz\frac{\partial^2}{\partial z \partial w}\ln E_1(z,w) \ , \tag{B.48}$$

and similarly for $\omega^{(2)}(z,w)$, where $E_1(z,w)$, $E_2(z,w)$ denote the prime forms on Σ_1, Σ_2 respectively. Integrating $\omega_j(z;t)$ in eq.(B.47) over the b cycles, we obtain the behavior of the period matrix $\Omega(t)$ of $\Sigma(t)$,

$$\Omega(t) = \begin{pmatrix} (\Omega_1)_{i_1 j_1} & 0 \\ 0 & (\Omega_2)_{i_2 j_2} \end{pmatrix} - i2\pi t \begin{pmatrix} 0 & \omega^{(1)}_{i_1}(p_1)\omega^{(2)}_{j_2}(p_2) \\ \omega^{(2)}_{i_2}(p_2)\omega^{(1)}_{j_1}(p_1) & 0 \end{pmatrix} + \mathcal{O}(t^2) \ , \tag{B.49}$$

where Ω_1 and Ω_2 denote the period matrices of Σ_1 and Σ_2. Substituting eqs.(B.47) and (B.49) into the definition (A.8) of the prime form yields

$$\begin{aligned} E(z_1,w_1) &\sim E_1(z_1,w_1) && \text{for } z_1,w_1 \in \Sigma_1' \ , \\ E(z_2,w_2) &\sim E_2(z_2,w_2) && \text{for } z_2,w_2 \in \Sigma_2 \ , \\ E(z_1,z_2) &\sim E_1(z_1,p_1)E_2(p_2,z_2)(-t)^{-\frac{1}{2}} && \text{for } z_1 \in \Sigma_1', \ \ z_2 \in \Sigma_2' \ . \end{aligned} \tag{B.50}$$

Plugging eq.(B.47) into eq.(A.23), we have

$$\Delta_{j_1} \sim \Delta^{(1)}_{j_1} + h_2 \int_{P_0}^{p_1} \omega^{(1)}_{j_1} \ , \qquad \Delta_{j_2} \sim \Delta^{(2)}_{j_2} - (h_2 - 1)\int_{P_0'}^{p_2} \omega^{(2)}_{j_2} \ , \tag{B.51}$$

where $\Delta^{(1)}$, $\Delta^{(2)}$ denote the vectors of Riemann constants of Σ_1, Σ_2 for the base points P_0, P_0' respectively. Here P_0' is an arbitrary point on Σ_2' and we will take it as the base point of the Abel-Jacobi map on Σ_2 throughout the subsequent analyses.

It is proved in Ref. [43] that the Arakelov Green's function $G^{\mathrm{A}}(z;w)$ on the degenerating surface Σ behaves as

$$\begin{aligned} G^{\mathrm{A}}(z_1,w_1) &\sim -2\left(\frac{h_2}{h}\right)^2 \ln|\tau| + G^{\mathrm{A}}_1(z_1,w_1) - \frac{h_2}{h}G^{\mathrm{A}}_1(z_1,p_1) - \frac{h_2}{h}G^{\mathrm{A}}_1(w_1,p_1) \ , \\ G^{\mathrm{A}}(z_2,w_2) &\sim -2\left(\frac{h_1}{h}\right)^2 \ln|\tau| + G^{\mathrm{A}}_2(z_2,w_2) - \frac{h_1}{h}G^{\mathrm{A}}_2(z_2,p_2) - \frac{h_1}{h}G^{\mathrm{A}}_2(w_2,p_2) \ , \\ G^{\mathrm{A}}(z_1,z_2) &\sim 2\frac{h_1 h_2}{h^2}\ln|\tau| + \frac{h_1}{h}G^{\mathrm{A}}_1(z_1,p_1) + \frac{h_2}{h}G^{\mathrm{A}}_2(z_2,p_2) \ , \end{aligned} \tag{B.52}$$

for $z_1,w_1 \in \Sigma_1'$ and $z_2,w_2 \in \Sigma_2'$, where τ is defined as

$$\tau \equiv t\left(2g^{\mathrm{A}(1)}_{p_1\bar{p}_1}\right)^{\frac{1}{2}}\left(2g^{\mathrm{A}(2)}_{p_2\bar{p}_2}\right)^{\frac{1}{2}} \ . \tag{B.53}$$

Here $G_1^{\rm A}(z_1;w_1)$, $G_2^{\rm A}(z_2;w_2)$ are the Arakelov Green's function with respect to the Arakelov metrics $g^{\rm A(1)}_{z_1\bar z_1}$, $g^{\rm A(2)}_{z_2\bar z_2}$ on Σ_1, Σ_2, respectively. Taking eq.(A.21) into account, we find that eq.(B.52) yields the asymptotic behavior of the Arakelov metric,

$$\begin{aligned} 2g^{\rm A}_{z_1\bar z_1} &\sim |\tau|^{2\left(\frac{h_2}{h}\right)^2}\, 2g^{\rm A(1)}_{z_1\bar z_1}\, e^{2\frac{h_2}{h}G_1^{\rm A}(z_1,p_1)} , \\ 2g^{\rm A}_{z_2\bar z_2} &\sim |\tau|^{2\left(\frac{h_1}{h}\right)^2}\, 2g^{\rm A(2)}_{z_2\bar z_2}\, e^{2\frac{h_1}{h}G_2^{\rm A}(z_2,p_2)} . \end{aligned} \tag{B.54}$$

Asymptotics of $Z^{\rm LC}$

Let us study the behavior of $Z^{\rm LC}$, using the expression (B.46). For this purpose, we have to know the asymptotic behavior of the constant c on the degenerating surface Σ. This can be obtained by substituting eq.(B.52) into the second relation in eq.(A.27) as follows:

$$c(t) \sim -2\frac{h_1h_2}{h^2}\ln|\tau| + \frac{h_1}{h}c_1 + \frac{h_2}{h}c_2 \tag{B.55}$$

as $t\to 0$, where c_1 is a constant defined by using $g^{\rm A(1)}_{z\bar z}$, $G_1^{\rm A}(z;w)$ and the Mandelstam mapping $\rho_1(z)$ on Σ_1 in the same way as c defined on Σ in eq.(A.25), and similarly for c_2 on Σ_2. Let $\alpha_{p_1}=-\alpha_{p_2}$ be the string-length of the intermediate propagator in the light-cone string diagram corresponding to the plumbing fixture:

$$\alpha_{p_1} = -\alpha_{p_2} = \sum_{r_2}\alpha_{r_2} = -\sum_{r_1}\alpha_{r_1} . \tag{B.56}$$

In deriving eq.(B.55), we have used

$$\begin{aligned} |\alpha_{p_1}|^2 &= \exp\left[-\sum_{I_1}G_1^{\rm A}(z_{I_1};p_1) + \sum_{r_1}G_1^{\rm A}(p_1;Z_{r_1}) + c_1\right] , \\ |\alpha_{p_2}|^2 &= \exp\left[-\sum_{I_2}G_2^{\rm A}(z_{I_2};p_2) + \sum_{r_2}G_2^{\rm A}(p_2;Z_{r_2}) + c_2\right] , \end{aligned} \tag{B.57}$$

and $|\alpha_{p_1}|^2 = |\alpha_{p_2}|^2$.

Combined with eq.(B.55), eq.(A.25) yields

$$\partial\rho(z_1) \sim \partial\rho_1(z_1) , \qquad \partial\rho(z_2) \sim \partial\rho_2(z_2) , \tag{B.58}$$

for $z_1\in\Sigma_1'$ and $z_2\in\Sigma_2'$, as $t\to 0$. These can also be obtained from the relations

$$\rho(z_1) \sim \rho_1(z_1) + \alpha_{m_1}\ln(-t)^{-\frac{1}{2}} + \sum_{r_2}\alpha_{r_2}\ln E_2(p_2,Z_{r_2}) ,$$

$$
\begin{aligned}
\rho(z_2) \;\sim\;& \rho_2(z_2) + \alpha_{m_2} \ln(-t)^{-\frac{1}{2}} + \sum_{r_1} \alpha_{r_1} \ln E_1(p_1, Z_{r_1}) \\
&+ 2\pi i \int_{P'_0}^{p_2} \omega^{(2)} \frac{1}{\operatorname{Im} \Omega_2} \left(\sum_{r_2} \alpha_{r_2} \operatorname{Im} \int_{P'_0}^{Z_{r_2}} \omega^{(2)} + \alpha_{m_2} \operatorname{Im} \int_{P'_0}^{p_2} \omega^{(2)} \right) \\
&- 2\pi i \int_{P_0}^{p_1} \omega^{(1)} \frac{1}{\operatorname{Im} \Omega_1} \left(\sum_{r_1} \alpha_1 \operatorname{Im} \int_{P_0}^{Z_{r_1}} \omega^{(1)} + \alpha_{m_1} \operatorname{Im} \int_{P_0}^{p_1} \omega^{(1)} \right) ,
\end{aligned} \tag{B.59}
$$

up to a purely imaginary constant, which follow from the definition (2.3) of $\rho(z)$ and the behaviors of ω_i, $E(z,w)$ and Ω_{ij} on the degenerating surface Σ. Eq.(B.58) yields

$$
\bar{N}^{r_1 r_1}_{00} \sim \bar{N}^{(1) r_1 r_1}_{\;\;00} , \qquad \bar{N}^{r_2 r_2}_{00} \sim \bar{N}^{(2) r_2 r_2}_{\;\;00} , \tag{B.60}
$$

as $t \to 0$, where $\bar{N}^{(1) r_1 r_1}_{\;\;00}$ and $\bar{N}^{(2) r_2 r_2}_{\;\;00}$ are zero-modes of the Neumann coefficients associated with the punctures Z_{r_1} and Z_{r_2} on the surfaces Σ_1 and Σ_2. Let us denote by $\rho(z_-)$ and $\rho(z_+)$ the interaction points where the intermediate propagator corresponding to the plumbing fixture interacts on Σ'_1 and Σ'_2 respectively, as is depicted in Figure 6. Using eq.(B.59), we find that $T_{\rm int} \equiv \operatorname{Re} \rho(z_+) - \operatorname{Re} \rho(z_-)$ is asymptotically related to t by

$$
\frac{T_{\rm int}}{\alpha_{m_2}} \sim - \ln |t| + \operatorname{Re} \bar{N}^{(1) p_1 p_1}_{\;\;00} + \operatorname{Re} \bar{N}^{(2) p_2 p_2}_{\;\;00} , \tag{B.61}
$$

where $\bar{N}^{(1) p_1 p_1}_{\;\;00}$, $\bar{N}^{(2) p_2 p_2}_{\;\;00}$ denote the Neumann coefficients associated with the punctures p_1, p_2 on Σ_1, Σ_2, respectively.

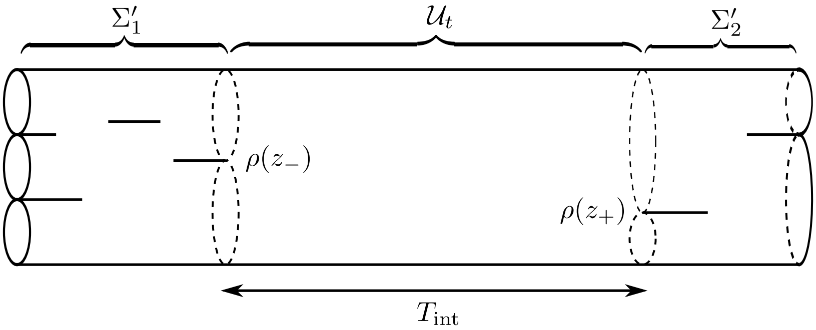

Figure 6: The degeneration of the string diagram described by the pluming fixture $\mathcal{U}_t$. The limit $t \to 0$ corresponds to the limit $T_{\rm int} \to \infty$.

The behavior of the Faltings' invariant $\delta(\Sigma)$ on the degenerating surface Σ can be deduced [43] by the use of eqs.(B.52) and (B.54) as

$$
e^{-\frac{1}{4}\delta(\Sigma)} \sim |\tau|^{\frac{h_1 h_2}{h}} e^{-\frac{1}{4}\delta(\Sigma_1)} e^{-\frac{1}{4}\delta(\Sigma_2)} . \tag{B.62}
$$

Gathering all the results obtained above, we eventually find that on the degenerating surface Z^{LC} factorizes as

$$Z^{\mathrm{LC}} \sim \frac{\mathcal{C}_{h,N}(\beta_A)}{\mathcal{C}_{h_1,N_1}(\beta_{A_1})\mathcal{C}_{h_2,N_2}(\beta_{A_2})}\, e^{\frac{2T_{\mathrm{int}}}{\alpha_{m_2}}}\, Z_1^{\mathrm{LC}} Z_2^{\mathrm{LC}}\ , \tag{B.63}$$

where Z_1^{LC} and Z_2^{LC} are the partition functions on Σ_1 and Σ_2 respectively. In order that Z^{LC} should correctly factorize as

$$Z^{\mathrm{LC}} \sim e^{\frac{2T_{\mathrm{int}}}{\alpha_{m_2}}}\, Z_1^{\mathrm{LC}} Z_2^{\mathrm{LC}}\ , \tag{B.64}$$

$\mathcal{C}_{h,N}(\beta_A)$ has to satisfy

$$\frac{\mathcal{C}_{h,N}(\beta_A)}{\mathcal{C}_{h_1,N_1}(\beta_{A_1})\mathcal{C}_{h_2,N_2}(\beta_{A_2})} = 1\ . \tag{B.65}$$

Repeating the same procedure, we can see that the evaluation of $\mathcal{C}_{h,N}$ reduces to that of $\mathcal{C}_{1,N}$.

B.2.2 $h = 1$ case

For $h = 1$, it is convenient to define a complex coordinate u on Σ such that

$$du = \omega\ , \tag{B.66}$$

where ω is the unique holomorphic one-form satisfying

$$\oint_a \omega = 1\,, \qquad \oint_b \omega = \tau\,. \tag{B.67}$$

In terms of the coordinate u and the moduli parameter τ, the prime form $E(u,u')$ takes the form

$$E(u,u') = \frac{\theta_1(u-u'|\tau)}{\theta_1'(0|\tau)}\ . \tag{B.68}$$

Here $\theta_1(u|\tau)$ denotes the theta function for $h = 1$ with the odd spin structure, defined as

$$\theta_1(u|\tau) \equiv -\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(u|\tau) = -e^{\frac{i}{4}\pi\tau + i\pi\left(u+\frac{1}{2}\right)}\theta\left(u+\frac{\tau}{2}+\frac{1}{2}\middle|\,\tau\right)\ , \tag{B.69}$$

which is related to the Dedekind eta function $\eta(\tau)$ by $\theta_1'(0|\tau) = 2\pi\eta(\tau)^3$. Accordingly, the Mandelstam mapping $\rho(u)$ becomes

$$\rho(u) = \sum_{r=1}^{N}\alpha_r\left[\ln\theta_1(u-U_r|\tau) - 2\pi i\frac{\operatorname{Im}U_r}{\operatorname{Im}\tau}(u-u_0)\right], \tag{B.70}$$

where U_r $(r = 1,\ldots,N)$ denote the punctures and u_0 denotes the base point on the u-plane. Let u_I $(I = 1,\ldots,N)$ be the interaction points on the u-plane, determined by $\partial\rho(u_I) = 0$.

For the $h = 1$ Riemann surface Σ that we are considering, eq.(A.22) tells us that there exist integers m and n such that

$$\sum_{I=1}^{N} u_I - \sum_{r=1}^{N} U_r = m + n\tau \ . \tag{B.71}$$

The Arakelov metric $g^{\mathrm{A}}_{u\bar{u}}$ does not depend on $u, \bar{u}$ because of eq.(A.16) and the Arakelov Green's function $G^{\mathrm{A}}(u; u')$ is given as

$$G^{\mathrm{A}}(u; u') = -\ln \left| \frac{\theta_1(u - u'|\tau)}{\theta_1'(0|\tau)} \right|^2 + \frac{2\pi}{\operatorname{Im}\tau} \left(\operatorname{Im}(u - u')\right)^2 - \ln(2g^{\mathrm{A}}_{u\bar{u}}) \, . \tag{B.72}$$

Substituting all these into eq.(B.46), we obtain

$$Z^{\mathrm{LC}} = \mathcal{C}_{1,N}(\beta)(2\pi)^{-16}(\operatorname{Im}\tau)^{-12} \left|\eta(\tau)\right|^{-48} \prod_{r=1}^{N} \left(e^{-2\operatorname{Re}\bar{N}^{rr}_{00}} \alpha_r^{-2} \right) \prod_{I=1}^{N} \left|\partial^2 \rho(u_I)\right|^{-1} \ . \tag{B.73}$$

$\mathcal{C}_{1,N}(\beta)$ can be fixed by considering the degeneration of the Riemann surface Σ of the type depicted in Figure 7. In this degeneration, two non-trivial cycles, the a cycle and its complement, are pinched to nodes p and p' respectively, and the genus 1 surface Σ will be divided into two disconnected spheres Σ_1 and Σ_2. We denote the nodes p, p' by p_1, p_1' or p_2, p_2' depending on whether they are regarded as punctures in Σ_1 or those in Σ_2. Similarly to the $h > 1$ case, we assume that the set of the punctures U_r $(r = 1, \ldots, N)$ are ordered so that in this degeneration they will be divided into two groups as $U_r = (U_{r_1}, U_{r_2})$ $(1 \leq r_1 \leq N_1,\ N_1 + 1 \leq r_2 \leq N)$ with $U_{r_1} \in \Sigma_1'$ and $U_{r_2} \in \Sigma_2'$. We may similarly assume that the set of the interaction points u_I $(I = 1, \ldots, N)$ will be divided as $u_I = (u_{I_1}, u_{I_2})$ $(1 \leq I_1 \leq N_1,\ N_1 + 1 \leq I_2 \leq N)$ with $u_{I_1} \in \Sigma_1'$ and $u_{I_2} \in \Sigma_2'$. The resultant spheres Σ_1 and Σ_2 are with $N_1 + 2$ punctures (U_{r_1}, p_1, p_1') and $N_2 + 2$ punctures (U_{r_2}, p_2, p_2'), respectively.

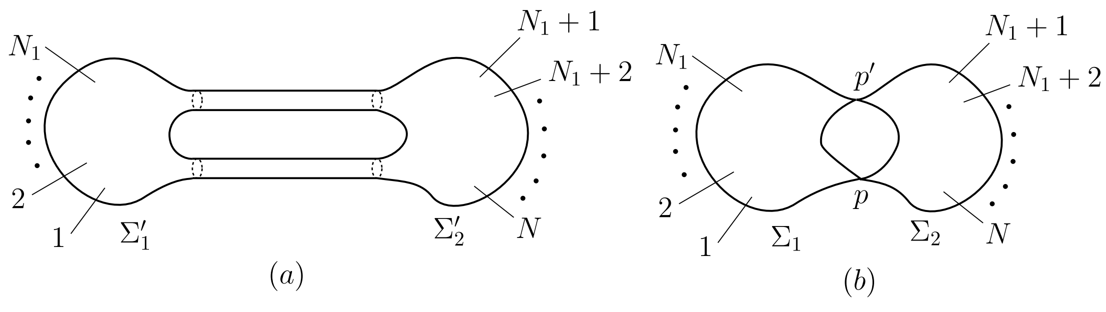

Figure 7: The degeneration of the $h = 1$ Riemann surface

The degeneration mentioned above is achieved by taking the limit in which the heights of the two cylinders corresponding to the internal propagators composing the loop are infinitely long (Figure 7 (a)). In the light-cone string diagram, this corresponds to the limit $T_{\rm int} \to \infty$ as depicted in Figure 8, which is the limit $\mathrm{Im}\,\tau \to \infty$ with the lengths of the two intermediate strings fixed. The length of one of the strings is proportional to

$$
\begin{aligned}
\oint_C du\, \partial\rho\,(u) &= \rho\,(u+1) - \rho\,(u) \\
&= -2\pi i \frac{\sum_r \alpha_r\, \mathrm{Im}\, U_r}{\mathrm{Im}\,\tau} + \sum_r (\pm\pi i)\, \alpha_r\,. \qquad \text{(B.74)}
\end{aligned}
$$

The limit we take is

$$
\begin{aligned}
U_{r_1}, u_{I_1} &\sim \mathcal{O}\left((\mathrm{Im}\,\tau)^0\right)\ , \\
U_{r_2} &= R\tau + U'_{r_2}\ , \qquad \text{with} \quad U'_{r_2} \sim \mathcal{O}\left((\mathrm{Im}\,\tau)^0\right)\ , \\
u_{I_2} &= R\tau + u'_{I_2}\ , \qquad \text{with} \quad u'_{I_2} \sim \mathcal{O}\left((\mathrm{Im}\,\tau)^0\right)\ , \qquad \text{(B.75)}
\end{aligned}
$$

where R is a real parameter such that $0 < R < 1$. We keep

$$
\sum_{r_1} \alpha_{r_1}\, \mathrm{Im}\, U_{r_1} + \sum_{r_2} \alpha_{r_2}\, \mathrm{Im}\, U'_{r_2} = 0 \qquad \text{(B.76)}
$$

in taking the limit $\mathrm{Im}\,\tau \to \infty$ to make

$$
\frac{\sum_r \alpha_r\, \mathrm{Im}\, U_r}{\mathrm{Im}\,\tau} = R \sum_{r_2} \alpha_{r_2} \qquad \text{(B.77)}
$$

fixed. Substituting eq.(B.75) into eq.(B.71) yields $n = 0$ and thus

$$
\sum_{I=1}^{N} u_I - \sum_{r=1}^{N} U_r = m\ , \qquad m \in \mathbb{Z}\ . \qquad \text{(B.78)}
$$

It is straightforward to show that in the degeneration limit $\mathrm{Im}\,\tau \to \infty$ addressed above, for $u \sim \mathcal{O}\left((\mathrm{Im}\,\tau)^0\right)$, defining

$$
z \equiv e^{-2\pi i u}\ , \qquad Z_{r_1} \equiv e^{-2\pi i U_{r_1}}\ , \qquad Z'_{r_2} \equiv e^{-2\pi i U'_{r_2}}\ , \qquad \text{(B.79)}
$$

we have

$$
\begin{aligned}
\rho(u) \sim\ & \rho_1(z) - \pi i R\tau \sum_{r_2} \alpha_{r_2} - \frac{1}{2}\sum_{r_1} \alpha_{r_1} \ln Z_{r_1} + \frac{1}{2}\sum_{r_2} \alpha_{r_2} \ln Z'_{r_2} \\
&+ \sum_{r_2} (\pm\pi i)\, \alpha_{r_2} + 2\pi i R u_0 \sum_{r_2} \alpha_{r_2}\ . \qquad \text{(B.80)}
\end{aligned}
$$

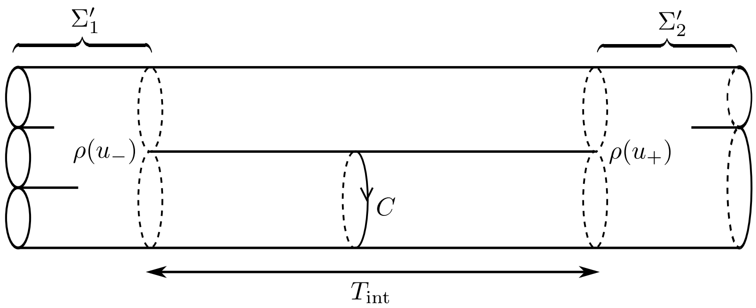

Figure 8: The degeneration of the light-cone string diagram. This corresponds to the limit $T_{\rm int} \to \infty$

Here $\rho_1(z)$ is defined as

$$\rho_1(z) \equiv \sum_{r_1=1}^{N_1} \alpha_{r_1} \ln(z - Z_{r_1}) + \left(R \sum_{r_2} \alpha_{r_2} \right) \ln z \ , \tag{B.81}$$

which coincides with the Mandelstam mapping on the sphere Σ_1 with parameters $Z_{p_1} = 0$, $\alpha_{p_1} = R \sum_{r_2} \alpha_{r_2}$ for puncture p_1 and $Z_{p'_1} = \infty$, $\alpha_{p'_1} = (R-1) \sum_{r_1} \alpha_{r_1}$ for puncture p'_1.

On the other hand, for $u = R\tau + u'$ with $u' \sim \mathcal{O}\left((\mathrm{Im}\,\tau)^0 \right)$, introducing

$$z' \equiv e^{-2\pi i u'} \ , \tag{B.82}$$

we obtain

$$\begin{aligned} \rho(u) \quad \sim \quad & \rho_2(z') - 2\pi i \left(R^2 - \frac{R}{2} \right) \tau \sum_{r_2} \alpha_{r_2} \\ & - \frac{1}{2} \sum_{r_1} \alpha_{r_1} \ln Z_{r_1} - \frac{1}{2} \sum_{r_2} \alpha_{r_2} \ln Z'_{r_2} + 2\pi i R u_0 \sum_{r_2} \alpha_{r_2} \ , \end{aligned} \tag{B.83}$$

where $\rho_2(z')$ is defined as

$$\rho_2(z') \equiv \sum_{r_2=1}^{N_2} \alpha_{r_2} \ln\left(z' - Z'_{r_2} \right) + \left((R-1) \sum_{r_2} \alpha_{r_2} \right) \ln z' \ , \tag{B.84}$$

which coincides with the Mandelstam mapping on the sphere Σ_2 with parameters $Z'_{p_2} = \infty$, $\alpha_{p_2} (= -\alpha_{p_1}) = R \sum_{r_1} \alpha_{r_1}$ for puncture p_2 and $Z'_{p'_2} = 0$, $\alpha_{p'_2} \left(= -\alpha_{p'_1} \right) = (R-1) \sum_{r_2} \alpha_{r_2}$ for puncture p'_2.

It follows from eqs.(B.80) and (B.83) that in the degeneration limit, the Neumann coefficients $\bar{N}^{r_1 r_1}_{00}$, $\bar{N}^{r_2 r_2}_{00}$ behave as

$$\bar{N}^{r_1 r_1}_{00} \sim \bar{N}^{(1) r_1 r_1}_{00} - \ln(-2\pi i) + 2\pi i U_{r_1} \ , \qquad \bar{N}^{r_2 r_2}_{00} \sim \bar{N}^{(2) r_2 r_2}_{00} - \ln(-2\pi i) + 2\pi i U'_{r_2} \ , \tag{B.85}$$

where $\bar{N}^{(1)r_1r_1}_{00}$, $\bar{N}^{(2)r_2r_2}_{00}$ denote the Neumann coefficients on Σ_1, Σ_2 associated with the punctures Z_{r_1}, Z'_{r_2} respectively, and that $\partial^2\rho(u_{I_1})$, $\partial^2\rho(u_{I_2})$ behave as

$$\partial^2\rho(u_{I_1}) \sim -(2\pi)^2 e^{-4\pi i U_{I_1}}\partial^2\rho_1(z_{I_1}) \ , \qquad \partial^2\rho(u_{I_2}) \sim -(2\pi)^2 e^{-4\pi i U'_{I_2}}\partial^2\rho_2(z'_{I_2}) \ , \tag{B.86}$$

where

$$z_{I_1} \equiv e^{-2\pi i u_{I_1}} \ , \qquad z'_{I_2} \equiv e^{-2\pi i u'_{I_2}} \ . \tag{B.87}$$

Let $\rho(u_-)$ and $\rho(u_+)$ be the interaction points in Σ'_1 and Σ'_2 on the light-cone string diagram where the long intermediate propagators interact as described in Figure 8. Using eq.(B.76), we can derive from eqs.(B.80) and (B.83) that $T_{\rm int} \equiv \mathrm{Re}\,\rho(u_+) - \mathrm{Re}\,\rho(u_-)$ is asymptotically related to $\mathrm{Im}\,\tau$ as

$$\left(\frac{1}{\alpha_{p_2}}+\frac{1}{\alpha_{p'_2}}\right)T_{\rm int} \sim 2\pi\,\mathrm{Im}\,\tau + \mathrm{Re}\left(\bar{N}^{(1)p_1p_1}_{00} + \bar{N}^{(1)p'_1p'_1}_{00} + \bar{N}^{(2)p_2p_2}_{00} + \bar{N}^{(2)p'_2p'_2}_{00}\right) \ , \tag{B.88}$$

where $\bar{N}^{(1)p_1p_1}_{00}$, $\bar{N}^{(1)p'_1p'_1}_{00}$ are the Neumann coefficients on Σ_1 associated with the punctures p_1, p'_1, and similarly for $\bar{N}^{(2)p_2p_2}_{00}$, $\bar{N}^{(2)p'_2p'_2}_{00}$ with the roles of Σ_1 and Σ_2 interchanged.

Gathering all the results obtained above and using the behavior $|\eta(\tau)|^{-48} \sim e^{4\pi\,\mathrm{Im}\,\tau}$ as $\mathrm{Im}\,\tau \to \infty$, we conclude that on the degenerating surface, the partition function $Z^{\rm LC}$ behaves as

$$Z^{\rm LC} \sim \mathcal{C}_{1,N}(\beta)\left(32\pi^2\right)^4\left(8\pi^2\,\mathrm{Im}\,\tau\right)^{-12}\exp\left(\frac{2T_{\rm int}}{\alpha_{p_2}}+\frac{2T_{\rm int}}{\alpha_{p'_2}}\right)Z_1^{\rm LC}Z_2^{\rm LC} \ , \tag{B.89}$$

where $Z_1^{\rm LC}$, $Z_2^{\rm LC}$ are the tree-level partition functions given in Ref. [11]

$$Z^{\rm LC} = \left|\sum_{s=1}^N \alpha_s Z_s\right|^4 \prod_r\left(e^{-2\,\mathrm{Re}\,\bar{N}^{rr}_{00}}\alpha_r^{-2}\right)\prod_I\left|\partial^2\rho(z_I)\right|^{-1} \tag{B.90}$$

for the surfaces Σ_1, Σ_2 with $Z_{p'_1} \to \infty$ for $Z_1^{\rm LC}$ and $Z'_{p_2} \to \infty$ for $Z_2^{\rm LC}$. The factor $\left(8\pi^2\,\mathrm{Im}\,\tau\right)^{-12}$ on the right hand side of eq.(B.89) coincides with that from the integration over the loop momentum and $\exp\left(\frac{2T_{\rm int}}{\alpha_{p_2}}+\frac{2T_{\rm int}}{\alpha_{p'_2}}\right)$ can be identified with the contribution from the tachyon mass. Therefore by taking

$$\mathcal{C}_{h=1,N}(\beta) = \frac{1}{(32\pi^2)^4} \ , \tag{B.91}$$

we get the factorization property as desired. Thus we find that $\mathcal{C}_{h=1,N}(\beta)$ is just a numerical constant. With this equation taken as initial condition, we can inductively solve eq.(B.65) for $\mathcal{C}_{h,N}(\beta_A)$ and obtain

$$\mathcal{C}_{h,N}(\beta_A) = \frac{1}{(32\pi^2)^{4h}} \ . \tag{B.92}$$

We eventually get $Z^{\rm LC}$ as is given in eq.(2.10).

C Modular transformations

In this appendix, we would like to show that the partition function $Z^{\rm LC}$ and the correlation function $\langle \prod_{r=1} V_r^{\rm LC} \rangle^{X^i}$ are modular invariant respectively.

We first review the modular properties of mathematical quantities on the surface Σ to fix the notations. Suppose that $M \in Sp(2h, \mathbb{Z})$, namely M is a $2h \times 2h$ integral matrix satisfying

$$MJM^T = J \ , \qquad J \equiv \begin{pmatrix} 0 & 1_h \\ -1_h & 0 \end{pmatrix} \ , \tag{C.1}$$

where 1_h denotes the $h \times h$ unit matrix. Decomposing the matrix M as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A, B, C, D being the $h \times h$ matrices, one can show that these matrices satisfy

$$\begin{cases} AD^T - BC^T = 1_h \\ AB^T = BA^T \\ CD^T = DC^T \end{cases} , \quad \begin{cases} A^TD - C^TB = 1_h \\ A^TC = C^TA \\ B^TD = D^TB \end{cases} , \quad \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in Sp(2h, \mathbb{Z}) \ . \tag{C.2}$$

Let us consider the modular transformation under which the homology basis $\{a_i, b_i\}$ $(i = 1, \ldots, h)$ transforms as

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \ . \tag{C.3}$$

Under this transformation, ω and Ω respectively transform as

$$\omega \mapsto \tilde{\omega} = \omega \frac{1}{C\Omega + D} \ , \qquad \Omega \mapsto \tilde{\Omega} = (A\Omega + B)\frac{1}{C\Omega + D} \ . \tag{C.4}$$

For the theta function with spin structure $[s] = \left[\begin{smallmatrix} s' \\ s'' \end{smallmatrix}\right]$, there is the following transformation law [45]:

$$\theta[\tilde{s}]\left(\tilde{\zeta}\middle|\tilde{\Omega}\right) = \varepsilon(M) e^{i\pi\phi(s)} \det\left(C\Omega + D\right)^{\frac{1}{2}} \exp\left(i\pi\zeta \frac{1}{C\Omega + D} C\zeta\right) \theta[s](\zeta|\Omega) \ , \tag{C.5}$$

where $\varepsilon(M)$ is an eighth root of unity depending on M,

$$\begin{aligned} &\tilde{s} \equiv \tilde{s}_0 + \delta \ , \qquad [\tilde{s}_0] = \begin{bmatrix} \tilde{s}'_0 \\ \tilde{s}''_0 \end{bmatrix} \ , \qquad [\delta] = \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \ , \\ &\begin{pmatrix} \tilde{s}'_0 \\ \tilde{s}''_0 \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} s' \\ s'' \end{pmatrix} \ , \qquad \begin{pmatrix} \delta' \\ \delta'' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathrm{diag}(CD^T) \\ \mathrm{diag}(AB^T) \end{pmatrix} \ , \end{aligned} \tag{C.6}$$

and

$$\tilde{\zeta} = \zeta \frac{1}{C\Omega + D} \ , \qquad \phi(s) = \tilde{s}'_0 \tilde{s}''_0 - s' s'' + 2\tilde{s}'_0 \delta'' \ . \tag{C.7}$$

The transformation law (C.5) leads to

$$\Delta \mapsto \tilde{\Delta} \equiv \Delta \frac{1}{C\Omega + D} + \delta' \tilde{\Omega} + \delta'' \qquad (\mathrm{mod}\ \mathbb{Z}^h + \mathbb{Z}^h \Omega) \ . \tag{C.8}$$

This is an immediate result of the relation,

$$\begin{aligned} \theta\left(\tilde{\zeta} + \delta'\tilde{\Omega} + \delta'' \middle| \tilde{\Omega}\right) &= \exp\left[-i\pi\delta'\tilde{\Omega}\delta' - i2\pi\zeta\frac{1}{C\Omega + D}\delta' + i\pi\zeta\frac{1}{C\Omega + D}C\zeta\right] \\ &\quad \times \varepsilon(M) e^{-i2\pi\delta'\delta''} \det\left(C\Omega + D\right)^{\frac{1}{2}} \theta(\zeta|\Omega) \ , \end{aligned} \tag{C.9}$$

which is obtained from eq.(C.5) by setting $s = 0$ and using eq.(A.3).

Using eq.(C.2), we can show that the matrices $\tilde{\Omega}$ and $\frac{1}{C\Omega + D}C$ are symmetric,

$$\mathrm{Im}\,\tilde{\Omega} = \frac{1}{\bar{\Omega}C^T + D^T}\,\mathrm{Im}\,\Omega \frac{1}{C\Omega + D} \ , \tag{C.10}$$

and thereby

$$\mathrm{Im}\left(v\frac{1}{C\Omega + D}\right)\frac{1}{\mathrm{Im}\,\tilde{\Omega}}\,\mathrm{Im}\left(\frac{1}{\Omega C^T + D^T}v\right) = \mathrm{Im}\,v\frac{1}{\mathrm{Im}\,\Omega}\,\mathrm{Im}\,v - \mathrm{Im}\left(v\frac{1}{C\Omega + D}Cv\right) \tag{C.11}$$

for an arbitrary vector $v \in \mathbb{C}^h$. This relation is useful in the following calculation.

Now let us show the modular invariance of $Z^{\mathrm{LC}} = Z^X[g^{\mathrm{A}}_{z\bar{z}}]^{24} e^{-\Gamma\left[g^{\mathrm{A}}_{z\bar{z}}, \ln|\partial\rho|^2\right]}$. We will show that each of $Z^X[g^{\mathrm{A}}_{z\bar{z}}]$ and $e^{-\Gamma\left[g^{\mathrm{A}}_{z\bar{z}}, \ln|\partial\rho|^2\right]}$ is modular invariant by itself. First we study the modular transformations of $e^{-\Gamma\left[g^{\mathrm{A}}_{z\bar{z}}, \ln|\partial\rho|^2\right]}$. Using eq.(C.5), one can find that the prime form $E(z,w)$ transforms as

$$E(z,w) \mapsto \tilde{E}(z,w) = \exp\left[i\pi\int_w^z \omega \frac{1}{C\Omega + D}C\int_w^z \omega\right] E(z,w) \ . \tag{C.12}$$

This yields that $F(z,\bar{z};w,\bar{w})$ defined in eq.(A.19) is modular invariant, and the Mandelstam mapping $\rho(z)$ given in eq.(2.3) just shifts by a factor independent of z as

$$\rho(z) \mapsto \tilde{\rho}(z) = \rho(z) + i\pi\sum_{r=1}^{N}\alpha_r \int_{P_0}^{Z_r} \omega \frac{1}{C\Omega + D} C \int_{P_0}^{Z_r} \omega \ . \tag{C.13}$$

This leads to the modular invariance of $\partial\rho(z)$ and $\bar{N}^{rr}_{00}$.[7] Taking into account the modular invariance of $F(z,\bar{z};w,\bar{w})$ and $\mu_{z\bar{z}}$ defined in eq.(A.12), one can show by the use of eqs.(A.16)

[7]We here assume that the local coordinate z is defined in a modular invariant way.

and (A.17) that $g^{\rm A}_{z\bar z}$ and $G^{\rm A}(z;w)$ are modular invariant. Now that we have found that $g^{\rm A}_{z\bar z}$, $G^{\rm A}(z;w)$, $\partial\rho(z)$ and $\bar N^{rr}_{00}$ are modular invariant, it is evident that $e^{-\Gamma\left[g^{\rm A}_{z\bar z},\,\ln|\partial\rho|^2\right]}$ given in eq.(B.41) is modular invariant as well.

Next, let us show that $Z^X[g^{\rm A}_{z\bar z}]$ is modular invariant. This is a direct result of the modular invariance of the Faltings' invariant $\delta(\Sigma)$ given in eq.(B.44), which can be seen as follows. For an arbitrary $\zeta\in\mathbb{C}^h$, the following relation holds:

$$
\begin{aligned}
& e^{-\pi\,{\rm Im}\left(\tilde\zeta+\tilde\Delta\right)\frac{1}{{\rm Im}\,\tilde\Omega}\,{\rm Im}\left(\tilde\zeta+\tilde\Delta\right)}\left|\theta\left(\tilde\zeta+\tilde\Delta\middle|\,\tilde\Omega\right)\right| \\
&\qquad = \left|\det\left(C\Omega+D\right)\right|^{\frac12} e^{-\pi\,{\rm Im}(\zeta+\Delta)\frac{1}{{\rm Im}\,\Omega}\,{\rm Im}(\zeta+\Delta)}\left|\theta\left(\zeta+\Delta\right|\Omega)\right|\ , \qquad \text{(C.14)}
\end{aligned}
$$

where $\tilde\zeta$ and $\tilde\Delta$ are defined in eqs.(C.7) and (C.8) respectively. This can be obtained by using eq.(C.9) with ζ replaced by $\zeta+\Delta$ and the relation

$$
\tilde\zeta+\tilde\Delta\equiv(\zeta+\Delta)\,\frac{1}{C\Omega+D}+\delta'\tilde\Omega+\delta'' \qquad (\mathrm{mod}\ \mathbb{Z}^h+\mathbb{Z}^h\Omega)\ . \qquad \text{(C.15)}
$$

Using eqs.(C.14), (C.4), (C.10), (A.5) and the modular invariance of $g^{\rm A}_{z\bar z}$ and $G^{\rm A}(z;w)$, one can find that $\delta(\Sigma)$ is modular invariant.

Finally we consider the modular invariance of the correlation function $\left\langle\prod_{r=1}^N V_r^{\rm LC}\right\rangle^{X^i}$. From the modular transformation law (C.13) of $\rho(z)$, one finds that the local coordinates w_r defined in eq.(2.6) is modular invariant. Eq.(C.13) also leads to the modular invariance of $\prod_r e^{-p_r^-\tau_0^{(r)}}$ contained in $\prod_r V_r^{\rm LC}$, in the presence of the delta-function $\delta\left(\sum_r p_r^-\right)$ responsible for the conservation of the momentum p^-. These imply that $\prod_r V_r^{\rm LC}$ in the correlation function is modular invariant. We also note that the scalar Green's function in the worldsheet theory can be described by using the worldsheet metric and modular invariant $F(z,\bar z;w,\bar w)$ [41]. Since we choose the modular invariant $g^{\rm A}_{z\bar z}$ as the worldsheet metric, the scalar Green's function is modular invariant as well. Thus we conclude that the correlation function $\left\langle\prod_{r=1}^N V_r^{\rm LC}\right\rangle^{X^i}$ is modular invariant.

Putting the results obtained above together, we find that the amplitudes (2.7) of the light-cone gauge string field theory in noncritical dimensions are modular invariant.

D A derivation of eq.(3.18)

In this appendix, we derive the identity (3.18):

$$
\prod_{r=1}^N\left(\alpha_r e^{2\,{\rm Re}\,\bar N^{rr}_{00}}\right)e^{-\Gamma\left[g^{\rm A}_{z\bar z},\,\ln|\partial\rho|^2\right]}Z^X[g^{\rm A}_{z\bar z}]^{-2}
$$

$$= \text{const.} \int \left[dbd\tilde{b}dcd\tilde{c} \right]_{g^{\mathrm{A}}_{z\bar{z}}} e^{-S^{bc}} \prod_{r=1}^{N} c\tilde{c}(Z_r, \bar{Z}_r) \prod_{K=1}^{6h-6+2N} \left[\int dz \wedge d\bar{z}\, i \left(\mu_K b + \bar{\mu}_K \tilde{b} \right) \right]. \quad \text{(D.1)}$$

In order to do so, we first rewrite the ghost path integral on the right hand side as follows. With the insertion of $\prod_{r=1}^{N} c\tilde{c}(Z_r, \bar{Z}_r)$, the integration over c and $\tilde{c}$ can be considered as the one over those $c, \tilde{c}$ which vanish at the punctures $z = Z_r$. Let ϕ_α ($\alpha = 1, \ldots, 3-3h+N$) be a basis of the holomorphic quadratic differentials on the punctured Riemann surface Σ which have no more than simple poles at the punctures Z_r. We decompose $b, \tilde{b}$ into the zero-modes and the nonzero-modes as

$$b(z,\bar{z}) = \sum_\alpha \phi_\alpha(z) b_\alpha + b'(z,\bar{z})\,, \qquad \tilde{b}(z,\bar{z}) = \sum_\alpha \bar{\phi}_\alpha(\bar{z}) \tilde{b}_\alpha + \tilde{b}'(z,\bar{z})\,. \quad \text{(D.2)}$$

The path integral measure $\left[dbd\tilde{b}dcd\tilde{c} \right]_{g^{\mathrm{A}}_{z\bar{z}}} \prod_{r=1}^{N} c\tilde{c}(Z_r, \bar{Z}_r)$ can be rewritten as

$$\left[dbd\tilde{b}dcd\tilde{c} \right]_{g^{\mathrm{A}}_{z\bar{z}}} \prod_{r=1}^{N} c\tilde{c}(Z_r, \bar{Z}_r) = \left[db'd\tilde{b}'dcd\tilde{c} \right]_{g^{\mathrm{A}}_{z\bar{z}}} \prod_\alpha \left(db_\alpha d\tilde{b}_\alpha \right) \prod_{r=1}^{N} c\tilde{c}(Z_r, \bar{Z}_r) J[g^{\mathrm{A}}_{z\bar{z}}]\,, \quad \text{(D.3)}$$

with the Jacobian factor $J[g^{\mathrm{A}}_{z\bar{z}}]$. Then the path integral on the right hand side of eq.(D.1) is equal to

$$\begin{aligned} & J[g^{\mathrm{A}}_{z\bar{z}}] \int \prod_\alpha \left(db_\alpha d\tilde{b}_\alpha \right) \prod_{K=1}^{6h-6+2N} \left[\sum_\alpha \int dz \wedge d\bar{z}\, i \left(\mu_K b_\alpha \phi_\alpha + \bar{\mu}_K \tilde{b}_\alpha \bar{\phi}_\alpha \right) \right] \\ & \quad \times \int \left[db'd\tilde{b}'dcd\tilde{c} \right]_{g^{\mathrm{A}}_{z\bar{z}}} e^{-S^{bc}[b',\tilde{b}',c,\tilde{c}]} \prod_{r=1}^{N} c\tilde{c}(Z_r, \bar{Z}_r)\,. \end{aligned} \quad \text{(D.4)}$$

If we consider a ghost path integral of the form

$$\int \left[dbd\tilde{b}dcd\tilde{c} \right]_{g^{\mathrm{A}}_{z\bar{z}}} e^{-S^{bc}} \prod_{r=1}^{N} c\tilde{c}(Z_r, \bar{Z}_r) \prod_{\alpha=1}^{3h-3+N} b\tilde{b}(S_\alpha, \bar{S}_\alpha)\,, \quad \text{(D.5)}$$

for arbitrary S_α ($\alpha = 1, \ldots, 3h-3+N$) instead, we can show that it is equal to

$$\begin{aligned} & J[g^{\mathrm{A}}_{z\bar{z}}] \int \prod_\gamma \left(db_\gamma d\tilde{b}_\gamma \right) \prod_{\beta=1}^{3h-3+N} \left[\sum_\alpha b_\alpha \phi_\alpha(S_\beta) \sum_\alpha \tilde{b}_\alpha \bar{\phi}_\alpha(\bar{S}_\beta) \right] \\ & \quad \times \int \left[db'd\tilde{b}'dcd\tilde{c} \right]_{g^{\mathrm{A}}_{z\bar{z}}} e^{-S^{bc}[b',\tilde{b}',c,\tilde{c}]} \prod_{r=1}^{N} c\tilde{c}(Z_r, \bar{Z}_r)\;. \end{aligned} \quad \text{(D.6)}$$

Therefore we can see that the right hand side of eq.(D.1) is rewritten as

$$\frac{\det\left(\int dz\wedge d\bar{z}\, i\mu_K\phi_\alpha\ ,\int dz\wedge d\bar{z}\, i\mu_K\bar{\phi}_\alpha\right)}{\left|\det\phi_\alpha(S_\beta)\right|^2}$$

$$\times\int\left[dbd\tilde{b}dcd\tilde{c}\right]_{g^{\mathrm{A}}_{z\bar{z}}} e^{-S^{bc}}\prod_{r=1}^{N}c\tilde{c}(Z_r,\bar{Z}_r)\prod_{\alpha=1}^{3h-3+N}b\tilde{b}(S_\alpha,\bar{S}_\alpha)\ . \qquad \text{(D.7)}$$

The bosonization formula [40, 41, 35, 21, 22] implies that eq.(D.5) is evaluated as

$$\begin{aligned}
&\int\left[dbd\tilde{b}dcd\tilde{c}\right]_{g^{\mathrm{A}}_{z\bar{z}}} e^{-S^{bc}}\prod_{r=1}^{N}c\tilde{c}(Z_r,\bar{Z}_r)\prod_{\alpha=1}^{3h-3+N}b\tilde{b}(S_\alpha,\bar{S}_\alpha)\\
&=\ Z^X[g^{\mathrm{A}}_{z\bar{z}}]\,(\det\mathrm{Im}\,\Omega)^{\frac{1}{2}}\left|\theta[\xi'](0|\Omega)\right|^2\prod_{r=1}^{N}\left(2g^{\mathrm{A}}_{Z_r\bar{Z}_r}\right)^{-1}\prod_{\alpha=1}^{3h-3+N}\left(2g_{S_\alpha\bar{S}_\alpha}\right)^2\\
&\qquad\times\exp\left[-\sum_{\alpha<\beta}G^{\mathrm{A}}(S_\alpha;S_\beta)-\sum_{r<s}G^{\mathrm{A}}(Z_r;Z_s)+\sum_{\alpha,r}G^{\mathrm{A}}(S_\alpha;Z_r)\right]\ ,
\end{aligned}\qquad \text{(D.8)}$$

up to a numerical multiplicative constant, where ξ' is given by

$$\xi'\equiv\sum_{\alpha=1}^{3h-3+N}\int_{P_0}^{S_\alpha}\omega-\sum_{r=1}^{N}\int_{P_0}^{Z_r}\omega-3\Delta\qquad(\mathrm{mod}\ \mathbb{Z}^h+\mathbb{Z}^h\Omega)\ . \qquad \text{(D.9)}$$

Now let us multiply the both sides of (D.8) by

$$\begin{aligned}
Z^X[g^{\mathrm{A}}_{z\bar{z}}]^3\ &=\ (\det\mathrm{Im}\,\Omega)^{-\frac{3}{2}}\left|\theta[\xi](0|\Omega)\right|^{-2}\left|\det\omega_j(\hat{z}_i)\right|^2\prod_{i=1}^{h}\left(2g^{\mathrm{A}}_{\hat{z}_i\bar{\hat{z}}_i}\right)^{-1}\\
&\qquad\times\exp\left[\sum_{i<j}G^{\mathrm{A}}(\hat{z}_i;\hat{z}_j)-\sum_{i}G^{\mathrm{A}}(\hat{z}_i;\hat{w})\right]\ ,
\end{aligned}\qquad \text{(D.10)}$$

derived from eq.(B.43). Here $\hat{z}_i$ $(i=1,\cdots h)$ and $\hat{w}$ can be arbitrarily chosen. As was done in Ref. [21], we take

$$\begin{aligned}
\hat{z}_i\ &=\ S_i\qquad\qquad(i=1,\cdots h)\,,\\
\hat{w}\ &=\ z_{2h-2+N}\ ,\\
S_{h+I'}\ &=\ z_{I'}\qquad\qquad(I'=1,\ldots,2h-3+N)\,,
\end{aligned}\qquad \text{(D.11)}$$

where z_I $(I=1,\cdots,2h-2+N)$ correspond to the interaction points of the light-cone diagram. Then ξ' given in eq.(D.9) equals to ξ given in eq.(B.45) and we find

$$\int\left[dbd\tilde{b}dcd\tilde{c}\right]_{g^{\mathrm{A}}_{z\bar{z}}} e^{-S^{bc}}\prod_{r=1}^{N}c\tilde{c}(Z_r,\bar{Z}_r)\prod_{i=1}^{h}b\tilde{b}\left(S_i,\bar{S}_i\right)\prod_{I'=1}^{2h-3+N}b\tilde{b}(z_{I'},\bar{z}_{I'})$$

$$
\begin{aligned}
&= Z^X [g^{\mathrm{A}}_{z\bar{z}}]^{-2} \frac{\left|\det \omega_j\left(S_i\right)\right|^2}{\det \operatorname{Im} \Omega} \left|\partial^2 \rho(z_{2h-2+N})\right|^{-2} \prod_{i=1}^{h} \left|\partial \rho\left(S_i\right)\right|^2 \\
&\quad \times e^{-2(h-1)c} \prod_{r=1}^{N} \left(\left(2 g^{\mathrm{A}}_{Z_r \bar{Z}_r}\right)^{-1} \left|\alpha_r\right|^{-1} \right) \prod_{I=1}^{2h-2+N} \left(2 g^{\mathrm{A}}_{z_I \bar{z}_I} \left|\partial^2 \rho(z_I)\right|\right) \ , \qquad \text{(D.12)}
\end{aligned}
$$

using eqs.(A.25), (A.27) and (A.28).

Following Ref. [20], let us take a basis $\phi_\alpha = (\phi_i, \phi_{h+I'})$ of the holomorphic quadratic differentials as

$$
\begin{aligned}
\phi_i &= d\rho\, \omega_i \qquad (i = 1, \ldots, h) \ , \\
\phi_{h+I'} &= d\rho\, \tilde{\omega}_{z_0 - z_{I'}} \qquad (I' = 1, \ldots, 2h-3+N) \ , \qquad \text{(D.13)}
\end{aligned}
$$

where $\tilde{\omega}_{P-Q}$ denotes the abelian differential of the third kind with simple poles at P and Q of residues 1 and -1 and with purely imaginary periods, which is given by

$$
\tilde{\omega}_{P-Q} = dz \partial_z \ln \frac{E(z,P)}{E(z,Q)} - 2\pi i \omega \frac{1}{\operatorname{Im} \Omega} \operatorname{Im} \int_Q^P \omega \ . \qquad \text{(D.14)}
$$

For the basis (D.13) we have

$$
\phi_i(z_{I'}) = 0 \ , \qquad \phi_{h+I'}(z_{J'}) = \delta_{I',J'}\, \partial^2 \rho(z_{I'}) \ , \qquad \text{(D.15)}
$$

and thus

$$
\det \phi_\alpha(S_\beta) = \det \omega_j(S_i) \prod_{I'=1}^{2h-3+N} \partial^2 \rho(z_{I'}) \prod_{i=1}^{h} \partial \rho(S_i) \ . \qquad \text{(D.16)}
$$

In Ref. [20], it is shown that

$$
\det \left(\int dz \wedge d\bar{z}\, i \mu_K \phi_\alpha \ , \int dz \wedge d\bar{z}\, i \bar{\mu}_K \bar{\phi}_\alpha \right) \propto \det \operatorname{Im} \Omega \ , \qquad \text{(D.17)}
$$

up to a numerical factor. Putting eqs.(D.7), (D.12), (D.16), (D.17) and (B.42) together, we obtain eq.(D.1).

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